PRZEGLĄD STATYSTYCZNY
R. LXII – ZESZYT 3 – 2015

STANISLAW MACIEJ KOT[1]

EQUIVALENCE SCALES BASED ON STOCHASTIC INDIFFERENCE CRITERION: THE CASE OF POLAND[2]

1. INTRODUCTION

The aim of the paper is to offer economists the stochastic equivalence scale (*SES*) as a new tool for making comparisons with respect to inequality, welfare and poverty in a heterogeneous population of households. The definition of the *SES* is 'axiomatic' in the sense that it only postulates the properties of a function that can be recognised as an *SES*. Any function can be considered an *SES* if and only if it transforms the distribution of expenditures of given groups of households in such a way that the resulting distribution is *stochastically indifferent* to the distribution of the expenditures of a reference group of households. Here, the property of stochastic indifference is a criterion of the homogeneity of transformed expenditure distributions. This criterion is also used to develop a method to estimate the *SES*.

This paper was motivated by serious deficiencies in recent solutions to the problem of addressing inequality, welfare and poverty when households differ in attributes other than their expenditures; e.g., other relevant differences include households' sizes and demographic compositions and household members' disabilities. When heterogeneity among the households exists, a two-step procedure has traditionally been applied. In the first step, a reference household group and an equivalence scale are chosen. Then, the actual expenditures for individual groups of households are adjusted by the equivalence scale (Buhmann et al., 1988; Jones, O'Donnell, 1995). In the second stage, standard measures of inequality, welfare, and poverty are applied to the adjusted distribution. These stages are treated separately.

However, there are two serious reasons why the two-stage procedure is unsatisfactory. First, the homogenisation stage does not provide unambiguous results. Second, the two stages seem to be interdependent.

The ambiguity of the first stage is because there is no single 'correct' equivalence scale for adjusting expenditures or incomes (Coulter et al., 1992a). Jäntti,

[1] Gdansk University of Technology, Department of Statistics, 11/12 Narutowicza St., 80-233 Gdansk, Poland, e-mail: skot@zie.pg.gda.pl.

[2] I am grateful to three anonymous referees for insightful comments. Funding from the National Centre of Science (NCN) Grant No. 2011/03/B/HS4/04962 is also gratefully acknowledged.

Danziger (2000, p. 319) remark that 'there is no optimal method for deriving an equivalence scale'. In fact, many serious identification issues arise in the estimation of equivalence scales (see, in particular, Pollak, Wales, 1979, 1992; Blundell, Lewbel, 1991; Blackorby, Donaldson, 1993 and the surveys of Lewbel, 1997, and Slesnick, 1998). Indeed, without additional assumptions, there is no way of selecting an appropriate basis with which to choose the equivalence scale. The *independence of base* (IB) (or the *exactness of equivalence* scale) is one such assumption. Several papers have tested this assumption, but they ultimately rejected it (Blundell, Lewbel, 1991; Blundell et al., 1998; Dickens et al., 1993; Pashardes, 1995; Gozalo, 1997; Pedankur, 1999).

If equivalence scales are arbitrary or if the IB assumption is not fulfilled, then the equivalent expenditure distribution cannot be unambiguously assigned to the chosen reference household group. Thus, the resulting population of fictitious equivalent units cannot be treated as homogeneous with respect to the selected attribute of households.

The second objection concerns the interdependence of the two stages of the traditional method of studying households (Ebert, Moyes, 2003). There is evidence that the results of distributional comparisons are sensitive to the choice of equivalence scale (Coulter et al., 1992a, b).

All of these problems with the equivalence-scale approach have encouraged economists to search for alternative solutions. Atkinson, Bourguignon (1987) propose the sequential Lorenz dominance approach for comparing the living standards in populations that include diverse incomes and needs. However, Ebert, Moyes (2003) note that this approach has only had limited empirical success; studies seem to favour the conventional equivalence scale for taking a family's circumstances into account.

Donaldson, Pendakur (2003) propose a method that is based on the concept of the *equivalent expenditure function*. Following this approach, Ebert, Moyes (2003) suggest a normative method for adjusting household incomes, and this method accounts for the heterogeneity of income recipients' needs when measuring inequality and welfare.

The stochastic equivalence scale transfers the problem of homogenisation from an individual person's perspective to the distributional level, i.e., the object of adjustment is the probability distribution of expenditures rather than an individual household's expenditures. The *SES* makes an initially heterogeneous population of households homogeneous with respect to such distributional features as social welfare, inequality and poverty.

The rest of this paper is organised as follows: Section 2 contains the definition of stochastic equivalence scales and section 3 explains the relationship between *SES*s and welfare, inequality and poverty. Section 4 introduces pragmatic equivalence scales, which can be considered potential *SESs*. Section 5 gives a statistical test to verify whether a particular function can be recognised as an *SES* and a method for estimating *SESs*. Section 6 contains the empirical results of estimating non-parametric and parametric *SESs* for Poland in the years 2005–2010. Lastly, section 7 concludes.

2. THE CONCEPT STOCHASTIC EQUIVALENCE SCALES

The following 'paradigmatic' issue that arises during welfare comparisons addresses the problem with equivalence scales: it is difficult to determine how much money ($\$x$) an m-person household would need to be as well off as a single person who spends $\$y$ each year. If the equivalence scale were of a relative type with a known deflator d, the answer to such a question would be $x = d{\cdot}y$. However, the arbitrariness of equivalence scales, which is discussed in the previous section, makes this equation unsolvable.

Donaldson, Pendakur (2003) propose the concepts of equivalent expenditure and an equivalence expenditure function for making welfare comparisons. Equivalent expenditure is the expenditure level which would make a single adult as well off as an m-member household; it may be written as a function of prices, expenditures and household characteristics. The corresponding equivalence scale is the actual expenditures divided by the equivalent expenditures. Similarly, the equivalent-expenditure function transforms the actual expenditures into equivalent expenditures. Donaldson, Pendakur (2003) maintain that: 'For welfare purposes, equivalent-expenditure functions permit the conversion of an economy with many household types into an economy of identical single individuals'. These authors show that, under certain conditions, equivalent-expenditure functions and their associated expenditure-dependent equivalence scales can be uniquely estimated from demand data. Following Donaldson and Pendakur's approach, Ebert, Moyes (2003) propose the concepts of equivalent income and the equivalent-income function.

The attractiveness of the equivalent-expenditure and equivalent-income concepts is due to the fact that their underlying assumptions are less demanding than the assumptions that are required by the equivalence-scale approach. However, some of these assumptions are not entirely convincing[3]. Although the new approach has resulted in many valuable theoretical achievements, some of the 'old' problems with equivalence scales still await solutions.

The concept of the stochastic equivalence scale (*SES*) offers an alternative potential solution to the problem of homogenising a heterogeneous population of households[4]. Let $h = [h_0,h_1,h_2,...,h_m]$ be the vector of an $m + 1$-level household attribute other than expenditures where $m + 1 > 2$, e.g., a household's size or its demographic composition would be a suitable attribute. Suppose that the population of all households (which will henceforth be called the total population) is divided into $m + 1$ subpopulations according to a certain attribute. Let the h_0-attribute subpopulation be chosen as the reference household group and let the continuous random variable Y with the distribution

[3] For example, Ebert, Moyes (2003) notice that the *Between-Type Transfer Principle* rules out using *utilitarianism* to make a relevant social judgement in their model. See also Capéau, Ooghe (2007).

[4] Early version of the SES was presented in Kot (2012). However, the link between the SES and stochastic indifference was not mentioned in that paper.

function $G(y)$ (which we abbreviate as $Y \sim G(y)$[5]) represent the expenditure distribution of this reference subpopulation. Henceforth, Y will be called 'the reference distribution'. Let the set of *m continuous* random variables $X_i \sim F_i(x)$, $i = 1,\ldots,m$, represent the expenditure distributions of the remaining m household subpopulations. From this point forwards, X_i will be called 'the evaluated distribution'.

Without loss of generality, the non-negative real-valued interval $[0,\infty)$ will be used as the domain of the considered random variables. However, the definition of the *SES* presented below would also be valid for all real numbers.

Suppose that $\boldsymbol{s}(\cdot) = [s_1(\cdot),\ldots,s_m(\cdot)]$ is a continuous and strictly monotonic real-valued vector function for which the inverse function $\boldsymbol{s}^{-1}(\cdot) = [s_1^{-1}(\cdot),\ldots,s_m^{-1}(\cdot)]$ exists, and suppose that this function is differentiable[6]. Let the random variable $Z_i = s_i(X_i) \sim H_i(z)$ be the transformation of the evaluated expenditure distribution X_i. Henceforth, the random variable $Z_i \sim H_i(z)$ will be called the 'transformed expenditure distribution'.

Denote by $Z \sim H(z)$ the following mixture of cumulative distribution functions $H_i(z)$:

$$H(z) = \sum_{i=1}^{m} \pi_i H_i(z), \text{ for all } z \geq 0, \tag{1}$$

where the weights π_i satisfy two conditions: $\forall i = 0,1,\ldots,m$, $\pi_i > 0$, and $\sum_{i=1}^{m} \pi_i = 1$. A single π_i weight can be interpreted as the proportionate size of the ith subpopulation relative to the size of the total population[7].

Definition 2.1. With the above notations, the function $\boldsymbol{s}(\cdot)$ will be called the *stochastic equivalence scale* (*SES*) if and only if the following equation holds:

$$\forall z \geq 0,\ H(z) = G(z). \tag{2}$$

When the function $\boldsymbol{s}(\cdot)$ is an *SES*, the transformed expenditure distributions Z will be called 'the equivalent expenditure distributions'.

We call the defined equivalence scales 'stochastic' to underline the fact that they transform random variables. 'Classical' equivalence scales can be called 'individualistic' because they transform individual expenditures (or incomes) in pairs.

The above definition of an *SES* is axiomatic in the sense that it only postulates the criterion for a function to be recognised as an *SES*. This definition does not describe how an *SES* should be constructed or the conditions of its existence. In other words, *any* function $\boldsymbol{s}(\cdot)$ that fulfils condition (2) has to be recognised as an *SES*.

5 We reserve capital letters for random variables and lowercase letters for the values that are taken by these variables.

6 These are standard conditions when transforming continuous random variables.

7 The number of persons or an equivalent unit is used when calculating π_i wages.

Naturally, the definition of an *SES* also applies when $m = 1$, i.e., when only one group of households is compared to the reference group. It is easy to see that condition (2) will automatically be fulfilled when the equation

$$H_i(z) = G(z) \tag{3}$$

holds for all $z \geq 0$ and for all $i = 1,...,m$. However, in general, equality (2) does not imply equality (3).

The relative *SES* can be defined as follows: let $\boldsymbol{d} = [d_i]$, $i = 1,...,m$, be the vector of positive numbers called 'deflators' that transform the evaluated expenditure distributions $X_1,...,X_m$ thusly:

$$Z_i = X_i/d_i \sim H_i(z),\ i = 1,...,m. \tag{4}$$

Definition 2.2. Under the above notations, the vector $\boldsymbol{d}$ will be called the *relative SES* if and only if the deflators $d_1,...,d_m$ are such that equality (2) holds.

The concept of the *SES* has several advantages that make it an interesting alternative to the equivalence scales that have been developed so far. Some of these advantages are discussed below.

The validation of condition (2) can be tested using nonparametric statistical tests for equality between cumulative distribution functions.

As potential *SESs*, parametric and nonparametric functions of $\boldsymbol{s}(\cdot)$ can easily be estimated on the basis of expenditure data[8]. Details of the statistical procedures that are used to test and estimate an *SES* are presented in section 5. It is worth adding that the usual procedure of extracting equivalence scales from the estimated model of a demand system is not necessary when estimating an *SES*.

Estimating an *SES* can potentially be useful when the relationship between household needs and socio-demographic attributes is ambiguous. It is generally accepted that the larger a household is, the greater its needs are. However, the classification of household types with respect to household needs may be problematic when more than a single attribute must be accounted for, e.g., a household including a single mother who has two children or a household with two parents and one disabled child.

The *SES* offers a solution to this problem. Assume that there are m groups of evaluated households that are selected with respect to certain criteria, but assume that these households are not necessarily selected with respect to their needs. Let the reference group also be selected. We can apply the nonparametric relative *SES* (4) with m deflators $d_1,...,d_m$. Let the estimates of these deflators be arranged in ascending order, i.e., $\widehat{d}_{(1)},\ldots,\widehat{d}_{(m)}$. Then, these deflators will rank the evaluated household groups according to their needs.

8 The parametric and nonparametric equivalence scales are defined in section 3.

3. THE RELATION BETWEEN AN *SES* AND WELFARE, INEQUALITY AND POVERTY

There are well known relations between stochastic dominance and economic inequality and between welfare and poverty (e.g., Davidson, 2008). In contrast, the concept of an *SES* satisfies the criterion of stochastic indifference as a symmetric factor of stochastic dominance. Hence, one can find the relation between an *SES* and the above-mentioned aspects of income or expenditure distributions.

Consider two non-negative[9] distributions of incomes or expenditures X_A and X_B and suppose that they are characterised by the cumulative distribution functions $F_A(x)$ and $F_B(x)$, respectively. Distribution X_B stochastically dominates distribution X_A in the first order if $F_A(x) \geq F_B(x)$ for any argument x (Davidson, 2008).

Higher orders of stochastic dominance can be defined in the following recursive way. Let $D^1(x) = F(x)$ and let $D^{s+1}(x) = \int_0^x D^s(t)dt$, for $s = 1,2,3,\ldots$ Distribution X_B dominates distribution X_A in order s if $D_A^s(x) \geq D_B^s(x)$ for all arguments x (Davidson, 2008).

It is easy to see that first-order stochastic dominance implies dominance in all higher orders. More generally, dominance in order s implies dominance in all orders higher than s (Davidson, 2008). However, it is not true that dominance in all orders higher than s implies dominance in order s.

Suppose that z is the poverty line in terms of incomes or expenditures. Then, $F(z)$ is the headcount ratio that measures the amount of poverty in a given distribution. The headcount ratio is higher in X_A than it is in X_B if and only if $F_A(x) > F_B(x)$ for all $x < z$.

We define the poverty gap for an individual with income x as $g(z,x) = max(z-x,0)$. Furthermore, we define the class of poverty indices over the poverty gaps as follows:

$$\Pi(z) = \int_0^z \pi(g(z,x))dF(x) \tag{5}$$

Atkinson (1987).

It can be shown that, in the case of all indices (5) where π is differentiable and $\pi(0) = 0$, $\Pi_A(x) \geq \Pi_B(x)$ occurs if and only if X_B stochastically dominates X_A up to z in the first order for all $x \leq z$ (Atkinson, 1987; Foster, Shorrocks, 1988; McFadden, 1989). This class of indices and the corresponding headcount ratio will be denoted as P^1. Similarly, class P^2 is defined by convex increasing functions π where $\pi(0) = 0$. It can be shown that all of the indices in P^2 are greater for X_A than for X_B if and only if X_B stochastically dominates X_A up to the second order for all $x \leq z$. In general, we can define class P^s in any order s such that it contain indices (5) with the following properties: $\pi^{(s)}(x) \geq 0$ for $x > 0$, $\pi^{(s-1)}(0) \geq 0$, and $\pi^{(i)}(0) = 0$ for $i = 0,\ldots,s-2$. Then, $\Pi_A(x) \geq \Pi_B(x)$ for all $x \leq z$ and for all $\Pi \in P^s$ if and only if X_B stochastically dominates X_A up to z in order s (Davidson, Duclos, 2000).

[9] We follow our restriction on the expenditure distributions' domain. In general, distribution functions are defined for all real arguments.

Stochastic dominance also allows welfare comparisons. Let U_1 denote the class of all von Neumann-Morgenstern-type utility functions u where $u' \geq 0$ (i.e., the function is increasing). Additionally, let U_2 denote the class of all utility functions in U_1 for which $u'' \leq 0$ (this condition implies strict concavity). Then, social welfare in distribution $X \sim F(x)$ can be defined as follows:

$$E[u(X)] = \int_0^\infty u(x)dF(x). \tag{6}$$

It can be seen that X_B implies that more social welfare will be awarded than does X_A if and only if X_A is stochastically dominated by X_B in the first order for all $u \in U_1$ and for all social welfare functions that have the form (6). When $u \in U_2$, all of the social welfare functions of this more restrictive class give an unambiguous ranking of two distributions if one dominates the other in the second order (Davidson, 2008).

A relation between stochastic dominance and inequalities can be obtained by means of Lorenz curves. Generalised Lorenz dominance is based on the generalised Lorenz curve (Shorrocks, 1983). Assuming $u \in U_2$, generalised Lorenz curves provide an unambiguous ranking of two distributions by means of the social welfare function (6). Generalised Lorenz dominance appears to be exactly the same as second-order stochastic dominance (Davidson, 2008).

All of these features of stochastic dominance also apply to stochastic indifference. We say that distribution X_A is stochastically indifferent to distribution X_B in the first order if and only if the following identity holds:

$$F_A(x) = F_B(x), \text{ for all } x \geq 0. \tag{7}$$

If we integrate both sides of identity (7) from 0 to x, we will obtain stochastic indifference in the second order, i.e.:

$$\int_0^x F_A(t)dt = \int_0^x F_B(t)dt, \text{ for all } x \geq 0. \tag{8}$$

Recursive integrations result in stochastic indifference in all higher orders. Thus, stochastic indifference in the first order implies that all higher orders are stochastically indifferent. Moreover, this implication also works in reverse. This fact can be shown by differentiating the integrals we have already obtained. Hence, stochastic indifference provides stronger results than stochastic dominance.

Condition (2) means that an *SES* guarantees the first-order stochastic indifference between the evaluated distribution and the reference distribution. This indifference implies that there is stochastic indifference at all higher orders, and at all lower orders.

All of the above considerations justify the following corollary:

Corollary 3.1.

Let X be the distribution of expenditures of the evaluated group of households, let Y be the distribution of expenditures of the reference group of households, and let $Z = s(X)$. If s is the corresponding *SES*, then the following equivalent conditions hold:

a) Z is stochastically indifferent to Y
b) Social welfare (6) in Z is exactly the same as in Y for all $u \in U_2$
c) Poverty in Z is exactly the same as poverty in Y for all poverty lines
d) Inequalities in Z are exactly the same as the corresponding inequalities in Y.

One may ask what type of homogeneity an *SES* provides. If an initial heterogeneous population of households consists of $m + 1$ subpopulations (including a reference subpopulation), then the adjustment of each m distinct expenditure distributions by the *SES* will result in new fictitious subpopulations that are homogeneous with respect to utilitarian social welfare, inequality and poverty.

Another advantage of the *SES* is related to Ebert, Moyes' (2003) opinion that the choice of equivalence scale and additional social judgements cannot be treated as two independent issues. Stochastic dominance appears as the base for the *SES* as well as for the measurement of inequality, social welfare, and poverty. Therefore, the *SES* links the problem of the homogenisation of a heterogeneous population of households with normative judgements.

4. PRAGMATIC EQUIVALENCE SCALES

Practitioners have developed various forms of equivalence scales, despite theoretical controversies (Coulter et al., 1992b). Indeed, from the theoretical point of view, these 'pragmatic' scales are arbitrary. However, even with this reservation, we can still treat them as potential *SES*s.

Several nonparametric and parametric pragmatic scales are in common use. Suppose that the total population of households is divided into $m + 1$ subpopulations and that one of them plays the role of a reference group. The nonparametric equivalence scale is the set $\{d_1,...,d_m\}$ of deflators (4), which are equal to the number of equivalent units that are attached to the ith group of households, $i = 1,...,m$. These deflators can be estimated by the method that is presented in section 5.

Jenkins, Cowell (1994) describe the parametric equivalence scale class as '…a set of scales sharing a common functional form and for which parametric variations change the scale rate relativities for households of a different type'. Following this definition, we will use the term *parametric SES* when the m deflators $d_1,...,d_m$ in equation (4) are a certain function of household attributes with several parameters. We denote this function as $d = d(\boldsymbol{h},\boldsymbol{\theta})$, where $\boldsymbol{h}$ is the vector of household attributes and $\boldsymbol{\theta}$ is the vector of the parameters. We denote the transformed distribution of expenditures as $Z = X / d(\boldsymbol{h},\boldsymbol{\theta})$, or equivalently, in the abbreviated form of $Z = X / d$, $Z \sim H(z)$, where X is the distribution of the expenditures of the evaluated group of households. According to definitions 2.1 and 2.2, this transformation will be the relative parametric *SES* if and only if $H(z) = G(z)$ for all $z \geq 0$, where $Y \sim G(y)$ describes the expenditure distribution of the reference group.

Certain forms of the parametric deflator $d(\boldsymbol{h},\boldsymbol{\theta})$ are especially popular in practical applications. The *power deflator* has the following form:

$$d = h^{\theta_1},\ 0 \leq \theta_1 \leq 1, \tag{9}$$

where h is the household size (Buhmann et al., 1988).

The parameter θ_1 is usually set arbitrarily. The per capita (with $\theta_1 = 1$) and square root (with $\theta_1 = 0.5$) equivalence scales appear to be the most popular equivalence scales (OECD, 2008).

Let a and k denote the numbers of adults and children, respectively. The *A-C* ('adults-children') deflator is defined as follows:

$$d = (a + \theta_1 k)^{\theta_2},\ \theta_1,\ \theta_2 > 0, \tag{10}$$

where θ_1, θ_2 are parameters (Coulter et al., 1992a, and independently, Cutler, Katz, 1992).

According to Cutler, Katz (1992), θ_1 is a constant reflecting the resource cost of a child relative to an adult. Parameter θ_2 reflects the overall economies of scale with respect to households' sizes. Jenkins, Cowell (1994) refer to $(a + \theta_1 k)$ as the 'effective household size'.

The *OECD-type* deflator can be defined as follows:

$$d = 1 + \theta_1 \cdot (a - 1) + \theta_2 \cdot k,\quad \theta_1, \theta_2 > 0, \tag{11}$$

where θ_1 and θ_2 are parameters.

The OECD scale assigns a value of 1 to the first adult and a value of θ_1 to every subsequent adult, whereas θ_2 is the weight that is attached to each child. The 'old OECD' scale (which is also referred to as the 'Oxford scale') assumes that $\theta_1 = 0.7$ and $\theta_2 = 0.5$ (OECD, 1982). This equivalence scale was in common use during the 1980s and early 1990s. In the late 1990s, the Statistical Office of the European Union (EUROSTAT) adopted the so-called OECD-modified (or augmented) equivalence scale, which assumes that $\theta_1 = 0.5$ and $\theta_2 = 0.3$. This scale was first proposed by Haagenars et al. (1994).

In addition, we can experiment with new versions of parametric scales. For instance, the logarithmic deflator is defined as follows:

$$d = 1 + \theta_1 \cdot \log h,\ \theta_1 > 0, \tag{12}$$

where h is the household size and θ_1 is the parameter to be estimated. The elasticity ε of this deflator with respect to the household size h is:

$$\varepsilon = \frac{1}{1/\theta_1 + \log h}. \tag{13}$$

We may note that this elasticity is a diminishing function of the household size.

We can treat the pragmatic equivalence scales as potential *SES*s. In our approach, these scales are estimated but not definitively designated. We will recognise them as *SES*s if condition (2) is fulfilled.

5. STATISTICAL ISSUES CONCERNING AN *SES*

There are two statistical issues related to *SESs*: testing whether a function $\mathbf{s}(\cdot)$ can be recognised as an *SES* and estimating parametric and non-parametric *SESs*. These two problems require random samples of expenditures per household. Micro-data are the most suitable data for this purpose, though grouped data may also be used.[10] We assume that the general population consists of $m + 1$ disjointed household populations and that each household represents a different type of household. One of these populations, which usually comprises one-person households, is treated as the reference population. The remaining m populations are called the evaluated populations.

In this section, we will use the notations and symbols defined in section 2. Whereas the random variable $Y \sim G(y)$ will represent the expenditure distribution in the general reference population, the expenditure distributions in the general evaluated populations will be represented by m random variables, $X_i \sim F_i(x)$, $i = 1,\ldots,m$. If the vector function $\mathbf{s}(\cdot) = [s_1(\cdot),\ldots,s_m(\cdot)]$ is a potential *SES,* then the random variables $Z_i = s_i(X_i) \sim H_i(z)$ will refer to the transformed income distributions in the general evaluated populations. The random variable $Z \sim H(z)$ will describe a mixture (1) of distributions Z_i in the general population and the symbols $\widehat{G}(y)$, $\widehat{F}_i(x)$, $\widehat{H}_i(z)$ and $\widehat{H}(z)$ will denote the sample distribution functions.

The random samples are defined as follows: the sample of size l that is from the general reference population will be denoted as $(y_1,\ldots,y_l)$. Whereas the random sample from the *i-th* evaluated general population will be denoted as $(x_1,\ldots,x_{n_i})_i$, the transformed version of this sample will be denoted as $(z_1,\ldots,z_{n_i})_i$, $i = 1,\ldots,m$. The total sample size of all of the evaluated groups will be denoted as $n = n_1 +,\ldots,+ n_m$ and weights $\pi_i = n_i / n$ are assigned to the ith group for each $i = 1,\ldots,m$. Let $(z_1,\ldots,z_n)$ denote the pooled sample of all of the transformed values with their corresponding weights $(\pi_1,\ldots,\pi_n)$.

We calculate the empirical distribution function $\widehat{G}(\cdot)$ for the reference distribution Y using the sample $(y_1,\ldots,y_l)$; similarly, we calculate the empirical distribution function $\widehat{H}(\cdot)$ using the pooled sample $(z_1,\ldots,z_n)$ and the corresponding weights $(\pi_1,\ldots,\pi_n)$.

[10] In this paper, we use micro-data when presenting statistical methods.

To verify that function $\boldsymbol{s}(\cdot) = [s_1(\cdot),\dots,s_m(\cdot)]$ is an *SES*, i.e., to check whether identity (2) holds, we need to test the null statistical hypothesis

$$\mathrm{H}_0\colon H(z) = G(z) \tag{14}$$

against the alternative hypothesis

$$\mathrm{H}_a\colon H(z) \neq G(z) \tag{15}$$

for all $z \geq 0$.

We will verify one of these hypotheses using the Kolmogorov-Smirnov (*K-S*) test:

$$U = \max_z |\hat{H}(z) - \widehat{G}(z)| \sqrt{\frac{l \cdot n}{l+n}}, \text{ for all } z \geq 0 \tag{16}$$

Smirnov (1939). Under the null hypothesis, the U statistic (16) has an asymptotic Kolmogorov-λ distribution (Kolmogorov, 1933).

The *p-value* of the *K-S* test (16), i.e., $p = P(U \geq u_{calc})$, is a convenient tool for testing these hypotheses on the selected significance level α, where u_{calc} is the calculated value of the U-test in the sample. If $p \leq \alpha$, we reject the null hypothesis H_0 and accept H_a, and as a result, the function $\boldsymbol{s}(\cdot)$ cannot be recognised as an *SES*. If $p > \alpha$, we accept the null hypothesis, and therefore, we recognise function $\boldsymbol{s}(\cdot)$ as an *SES*.

The proposed method of estimating *SESs* uses the U-test as a loss function. Suppose that the function $\boldsymbol{s}(\cdot)$ is a potential *SES*. This function may be non-parametric, e.g., it may take the form of a set of deflators $\boldsymbol{d} = [d_1,\dots,d_m]$, or it may be parametric and depend on k parameters $\boldsymbol{\theta} = [\theta_1,\dots,\theta_k]$. We will use the symbols $\boldsymbol{s}(\cdot|\boldsymbol{d})$, $\boldsymbol{s}(\cdot|\boldsymbol{\theta})$ or simply $\boldsymbol{s}$ when the context of the estimation is obvious. Let $z_1,\dots,z_n$ be the sequence of the evaluated expenditures that are adjusted by the function $\boldsymbol{s}$, i.e., let $z_j = \boldsymbol{s}(x_j|\boldsymbol{d})$ or $z_j = \boldsymbol{s}(x_j|\theta)$, $j = 1,\dots,n$. Let $\widehat{G}(z)$ and $\widehat{H}(z\,|\,\mathbf{s})$ denote the empirical distribution functions of the reference expenditures and the adjusted expenditures, respectively. We propose the estimator $\boldsymbol{s}^*$ of $\boldsymbol{s}$ that minimises the *K-S* statistic $U(\boldsymbol{s})$, i.e.:

$$U(\mathbf{s}^*) = \min_s \max_z |\widehat{H}(z\,|\,\mathbf{s}) - \widehat{G}(z)| \sqrt{\frac{n \cdot l}{n+l}}, \text{ for all } z \geq 0. \tag{17}$$

When we compare the reference distribution Y with a particular evaluated distribution Z_i, we can substitute for the terms $\widehat{H}(z\,|\,\mathbf{s})$ and n on the right-hand side of equation (17) with $\widehat{H}_i(z\,|\,\mathbf{s})$ and n_i, respectively, for each $i = 1,\dots,m$.

Alternatively, we can estimate the *SES* by using *p-values* as a function of $\boldsymbol{s}$, i.e., by using $p(\boldsymbol{s})$:

$$p(\boldsymbol{s}) = P[U(\boldsymbol{s}) \geq u_{calc}(\boldsymbol{s})]. \tag{18}$$

This *SES* estimator is $\boldsymbol{s}^*$ and it maximises the *p-value* (18):

$$p(\boldsymbol{s^*}) = \max_{\mathbf{s}} P[U(\boldsymbol{s}) \geq u_{calc}(\boldsymbol{s})] \wedge p(\boldsymbol{s^*}) > \alpha. \quad (19)$$

However, it should be noted that $\boldsymbol{s^*}$ estimators do not always exist. Although $p(\boldsymbol{s})$ usually reaches a maximum, the condition $p(\boldsymbol{s^*}) > \alpha$ might be violated. In practice, the estimator $\boldsymbol{s^*}$ can be found using the grid-search method.

6. STOCHASTIC EQUIVALENCE SCALES FOR POLAND 2005–2010

In this section, we will use expenditure distributions to estimate the *SESs*. The monthly micro-data come from the Polish Household Budget Surveys for the years 2005–2010. The expenditures are expressed in constant 2010 prices. The data were collected annually by the Central Statistical Office of Poland. We assume a 5% significance level in all of the analysed cases.

Table 1 presents estimates of nonparametric relative *SESs* (4) when the household groups are distinguished according to the number of members (i.e., according to the households' sizes). The deflators d_h are estimated separately for each household group.

Table 1

The estimated deflators of the non-parametric *SESs* for various household sizes. (95% confidence intervals in parentheses)

Household size	2005	2006	2007	2008	2009	2010
1	1	1	1	1	1	1
2	1.625 (1.601;1.656) $p = 0.704$	1.616 (1.605;1.628) $p = 0.180$	1.679 (1.670;1.700) $p = 0.383$	1.709 (1.692;1.733) $p = 0.557$	1.681 (1.662;1.721) $p = 0.611$	1.686 (1.662;1.728) $p = 0.618$
3	1.889 (1.879;1.900) $p = 0.324$	1.942 (*;*) $p = 0.027$	2.048 (2.024;2.081) $p = 0.603$	2.115 (2.095;2.158) $p = 0.356$	2.071 (2.055;2.121) $p = 0.263$	2.068 (2.062;2.093) $p = 0.090$
4	2.045 (2.003;2.070) $p = 0.303$	2.077 (2.069;2.088) $p = 0.118$	2.242 (2.211;2.295) $p = 0.648$	2.293 (2.285;2.309) $p = 0.111$	2.250 (2.232;2.283) $p = 0.202$	2.217 (2.202;2.276) $p = 0.228$
5	2.045 (2.003;2.007) $p = 0.303$	2.111 (2.106;2.115) $p = 0.061$	2.260 (2.260;2.270) $p = 0.065$	2.334 (2.324;2.347) $p = 0.096$	2.275 (*,*) $p = 0.007$	2.266 (*;*) $p = 0.006$
6 or more	2.179 (2.173;2.197) $p = 0.104$	2.303 (*;*) $p = 0.008$	2.474 (*;*) $p = 0.038$	2.485 (*,*) $p = 0.022$	2.466 (*;*) $p = 0.006$	2.543 (*;*) $p = 0.007$
p (K-S)	0.54071	0.07230	0.29703	0.06393	0.07053	0.03726
p (K-W)	0.82307	0.93336	0.89588	0.63157	0.83789	0.93808

Note: p(K-S): p-value in Kolmogorov-Smirnov test; *p(K-W)*: p-value in Kruskal-Wallis test.
Source: Polish Household Budget Survey, 2005–2010, own calculations.

An analysis of the results presented in table 1 shows that almost all of the estimated deflators can be recognised as *SESs*. The exceptions are two estimates for sizable household groups (specifically, households with five or more members). The second row from the bottom in table 1 shows that these exceptions influence the overall *K-S* test (15) only for the year 2010. However, the p-values of the overall Kruskal-Wallis test, which is presented in the last row of table 1, are greater than the significance level $\alpha = 0.05$. Thus, all of the estimated deflators can be recognised as *SESs*.

Three features of the estimated equivalence scales are remarkable. First, these nonparametric equivalence scales are very flat in comparison with the per capita scale. As a result, Polish households exhibited large economies of scale in the years 2005–2010. Second, the 95% confidence intervals are very narrow for the estimated deflators. Consequently, the proposed method to estimate the non-parametric scales is quite accurate. Third, equivalence scales vary over time.

Table 2 contains the estimates of the one-parameter pragmatic scales, i.e., the power scale (9) (Buhmann et al., 1988) and the logarithmic scale (12).

Table 2

The estimates of the one-parameter *SESs* (95% confidence intervals in parentheses)

Year	*Power:* $d = h^{\theta_1}$		*Logarithmic:* $d = 1 + \theta_1 \log h$	
	θ_1	*p-value*	θ_1.	*p-value*
2005	0.51872 (0.51267; 0.52559)	0.38150	0.75681 (0.73640; 0.77773)	0.61788
2006	0.53853 (0.53661;0.54090)	0.11294	0.79389 (0.78919; 0.80228)	0.11506
2007	0.58891 (0.58230; 0.59701)	0.47660	0.89671 (0.88455; 0.92414)	0.37624
2008	0.61021 (0.60595; 0.62193	0.24119	0.93199 (0.92955; 0.93490)	0.070294
2009	0.59684 (0.59176; 0.60856)	0.24987	0.90404 (0.90101; 0.91010)	0.071374
2010	0.59454 (0.59143; 0.60220)	0.14223	0.90424 (*; *)	0.03389

Note: h – household size.

Source: Polish Household Budget Survey, 2005–2010, own calculations.

Analysis of the results in table 2 shows that all of the power equivalence scales can be recognised as *SESs* because all of the corresponding *p-values* are greater than the significance level of 0.05. The estimates of θ_1 are less than one in every year under consideration. This is an indication of the economies of scale that were enjoyed by

Polish households in the years 2005–2010. However, the effect of economies of scale seems to diminish because the parameter θ_1 slowly increases in this period. It is also noteworthy that the value $\theta_1 = 0.5$ of power scale (9) is outside of the 95% confidence interval. Thus, a widely used 'square-root' pragmatic scale cannot be recognised as an *SES* of Polish households in this period.

The estimates of the logarithmic scale satisfy the *SES* condition because the *p-values* are greater than 0.05 for all of the years except 2010. This 'experimental' scale convinces us that the class of pragmatic equivalence scales could be enriched using new, more flexible forms.

Table 3 presents estimates of two two-parameter pragmatic scales: *A-C* (10) and OECD-Type (11). For comparison, the last two columns contain *p-values* of the *K-S* test (16) for two OECD scales, namely, 'old' and 'augmented', and their parameters θ_1 and θ_2 are not estimated, but they are arbitrarily designated.

Table 3

The estimates of the two-parameter *SESs*

Year	*Adults-Children:* $d = (a + \theta_1 k)^{\theta_2}$			*OECD-Type:* $d = 1 + \theta_1 (a - 1) + \theta_2 k$			*OECD 'old'*	*OECD augmented*
	θ_1.	θ_2	*p-value*	θ_1	θ_2	*p-value*	*p-value*	*p-value*
2005	0.48636	0.58616	0.76538	0.47261	0.14523	0.53367	0.00000	0.00000
2006	0.58051	0.59131	0.21795	0.49394	0.15152	0.17369	0.00000	0.00000
2007	0.61263	0.63990	0.58117	0.47035	0.32487	0.42113	0.00000	0.03268
2008	0.61080	0.6680	0.16365	0.53568	0.28392	0.32424	0.00000	0.00037
2009	0.61764	0.64322	0.12233	0.50327	0.29312	0.27139	0.00000	0.26723
2010	0.64658	0.63789	0.08588	0.48417	0.32362	0.21231	0.00000	0.14914

Note: *a* – the number of adults, *k* – the number of children under the age of 18, OECD 'old': $\theta_1 = 0.7$, $\theta_2 = 0.5$, OECD-augmented: $\theta_1 = 0.5$, $\theta_2 = 0.3$.

Source: Polish Household Budget Survey, 2005–2010, own calculations.

The estimates of the parameter θ_1 of the *A*-C equivalence scale show that the resource cost of a child in proportion to an adult increases from approximately 49% in 2005 to 65% in 2010. The elasticity of this scale with respect to the 'effective household size', i.e., $a + \theta_1 k$, increased in 2005–2008 and then decreased in the two subsequent years that were examined.

The parameters of the estimated OECD-type scales also vary over time. The weights that are assigned to the second and subsequent adults in a household fluctuate but remain close to 0.5. The weights assigned by this scale to each child under the age of 18 tend to increase from 0.15 to 0.32. In general, these estimates differ from the parameters of both the 'old' and the 'augmented' OECD scales. It is worth noting that the 'old' OECD scale cannot be recognised as an *SES* for all of the years under consideration, whereas the 'augmented' OECD scale can be recognised as an *SES* only for the years 2009 and 2010.

7. CONCLUSIONS

Thus far, the individualistic paradigm of consumer behaviour theory has been unable to solve the problem of homogenising a population of households that differ in all respects other than their expenditures. The adjustment of the expenditures of individual households fails when it is based on a pairwise equalisation of household utilities. Moreover, the separation of the adjustment procedure from the normative evaluations of welfare, inequality and poverty makes the pairwise equalisation quite untrustworthy.

The stochastic equivalence scale addresses the problem of homogenisation at the 'distributional' level, where only the probability distributions of expenditures are objects of adjustment, and therefore, individual household expenditures are not adjusted. An *SES* makes an initially heterogeneous population of households homogeneous with respect to such distributional features as social welfare, inequality and poverty. We use the term 'distributional' because social welfare, inequality and poverty indices characterise the distribution of expenditures but not of individuals. For these reasons, our equivalence scales are called 'stochastic'.

The axiomatic formulation of the *SES* is quite general. It does not specify one definite form of the scale, but it does define the properties that should be satisfied by a certain function for it to be recognised as an *SES.* The validation of these properties can be verified using statistical tests. The possibility of estimating parametric and non-parametric *SES*s also opens new perspectives for experimenting with other forms of such scales.

It should be emphasised that the actual form of an SES function is not important; only the fact that a function is an *SES* function matters. Thus, we do not have to search for an optimal *SES*. For instance, regardless of whether the *SES* is a power scale, an *A-C* scale or an OECD-type scale, it will always provide the same equivalent distribution of expenditures. Obviously, each of these scales (as well as other scales) can shed additional light on an evaluated distribution. For instance, an *A-C* or OECD-type scale can be useful in evaluating the cost of children.

It is worth noting that stochastic equivalence scales open new perspectives for comparisons of welfare, inequality and poverty between various geographic regions and/or periods. For this purpose, we need to choose one common reference group of households and express all of the distributions of expenditures in terms of comparable equivalence units.

The application of *SESs* is easy in practice. Estimating parametric and nonparametric *SESs* requires typical statistical data and standard statistical packages.

However, we encountered an unexpected problem when we applied an *SES* to the disposable income distributions in Poland in the years 2005–2010. Although we obtained plausible estimates of parametric and nonparametric equivalence scales, *none* of these scales could be recognised as *SES*s. Further research will be required to explain this occurrence.

REFERENCES

Atkinson A. B., (1987), On the Measurement of Poverty, *Econometrica,* 55, 749–764.

Atkinson A. B., Bourguignon F., (1987), Income Distribution and Differences in Needs, in: Feiwel G. R., (ed.), *Arrow and the Foundations of the Theory of Economic Policy*, Macmillan, London.

Blackorby C., Donaldson D., (1993), Adult-Equivalence Scales and the Economic Implementation of Interpersonal Comparisons of Well-Being, *Choice and Welfare*, 10, 335–361.

Blundell R. W., Duncan A., Pendakur K., (1998), Semiparametric Estimation of Consumer Demand, *Journal of Applied Econometrics*, 13, 435–461.

Blundell R. W., Lewbel A., (1991), The Information Content of Equivalence Scales, *Journal of Econometrics*, 150, 49–68.

Buhmann B., Rainwater L., Schmaus G., Smeeding T., (1988), Equivalence Scales, Well-Being, Inequality, and Poverty: Sensitivity Estimates Across Ten Countries Using the Luxembourg Income Study (LIS) Database, *Review of Income and Wealth*, 34, 115–142.

Capéau B., Ooghe E., (2007), On Comparing Heterogeneous Populations: Is There Really a Conflict Between Welfarism and a Concern for Greater Equality in Living Standards? *Mathematical Social Sciences*, 53, 1–28.

Coulter F. A. E., Cowell F. A., Jenkins S. P., (1992a), Differences in Needs and Assessment of Income Distributions, *Bulletin of Economic Research*, 44, 77–124.

Coulter F. A. E., Cowell F. A., Jenkins S. P., (1992b), Equivalence Scale Relativities and the Extent of Inequality and Poverty, *Economic Journal*, 102, 1067–1082.

Cutler A. E., Katz L. F., (1992), Rising Inequality? Changes in the Distribution of Income and Consumption in the 1980s, *Economic Review*, *Papers and Proceedings* 82, 546–551.

Davidson R., (2008), Stochastic Dominance, in: Durlauf S. N., Lawrence E. B., (eds.), *The New Palgrave Dictionary of Economics*, Second Edition, Palgrave Macmillan.

Davidson R., Duclos J., (2000), Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality, *Econometrica*, 68, 1435–1464.

Dickens R., Fry V., Pashardes P., (1993), Nonlinearities, Aggregation and Equivalence Scales, *Economic Journal*, 103, 359–368.

Donaldson D., Pendakur K., (2003), Equivalent-Expenditure Functions and Expenditure-Dependent Equivalence Scales, *Journal of Public Economics*, 88, 175–208.

Ebert U., Moyes P., (2003), Equivalence Scales Reconsidered, *Econometrica*, 71, 319–343.

Foster J. E., Shorrocks A. F., (1988), Poverty Orderings, *Econometrica*, 56, 173–177.

Gozalo P., (1997), Nonparametric Bootstrap Analysis with Implementation to Demographic Effects in Demand Functions, *Journal of Econometrics*, 81, 357–393.

Haagenaars A, de Vos K., Zaidi A., (1994), *Poverty Statistics in the Late 1980s, Research Based on Micro-data*, Office for Official Publications of the European Communities, Luxembourg.

Jones A., O'Donnell O., (1995), Equivalence Scales and the Costs of Disability, *Journal of Public Economics*, 56, 273–289.

Jäntti M., Danziger S., (2000), Poverty in Advanced Countries, in: Atkinson A. B., Bourguignon F., (eds.), *Handbook of Income Distribution*, North-Holland, Amsterdam.

Jenkins S. P., Cowell F. A., (1994), Parametric Equivalence Scales and Scale Relativities, *The Economic Journal*, 104, 891–900.

Kolmogorov A. N., (1933), Sulla Determinazione Empirica di Una Legge di Distribuzione, *Giornale delli Insitute Italiano degli Attuari*, 4, 83–91.

Kot S. M. (2012), *Towards the Stochastic Paradigm of Welfare Economics*, (in Polish: *Ku stochastycznemu paradygmatowi ekonomii dobrobytu*), Impuls, Cracow.

Kruskal W. H., Wallis W. A., (1952), Use of Ranks in One-Criterion Variance Analysis, *Journal of the American Statistical Association*, *47*, 583–621.

Lewbel A., (1997), Consumer Demand Systems and Household Equivalence Scales, in: Pesaran M. H., Schmidt P., (eds.), *Handbook of Applied Econometrics*, Volume II, *Microeconomics*, Blackwell Publishers Ltd, Oxford.

McFadden D., (1989), Testing for Stochastic Dominance, in: Fromby T. B., Seo T. K., (eds.), *Studies in the Economics of Uncertainty*, Springer-Verlag, New York.

OECD, (1982), *The List of Social Indicators*, Paris.

OECD, (2008), *Growing Unequal? Income Distribution and Poverty in OECD Countries*, Paris (www.oecd.org/els/social/inequality / www.oecd.org/els/social/inegalite).

Pashardes P., (1995), Equivalence Scales in a Rank-3 Demand System, *Journal of Public Economics*, 58, 143–158.

Pendakur K., (1999), Estimates and Tests of Base-Independent Equivalence Scales, *Journal of Econometrics*, 88, 1–40.

Pollak R. A., Wales T. J., (1979), Welfare Comparisons and Equivalence Scales, *American Economic Review*, 69, 216–221.

Pollak R. A., Wales T. J., (1992), *Demand System Specification and Estimation,* Oxford University Press, London.

Shorrocks A. F., (1983), Ranking Income Distributions, *Economica*, 50, 3–17.

Shorrocks A. F., (1998), Deprivation Profiles and Deprivation Indices, Jenkins S. P., Kapteyn A., van Praag B. M. S., (eds.), Ch. 11 in *The Distribution of Household Welfare and Household Production*, Cambridge University Press.

Slesnick D., (1998), Empirical Approaches to the Measurement of Welfare, *Journal of Economic Literature*, 36, 2108–2165.

Smirnov N. W., (1939), Sur les Ecarts de la Courbe de Distribution Empirique, *Comptes Rendus de l'Academie des Sciences Paris*, 6, 3–26.

SKALE EKWIWALENTNOŚCI BAZUJĄCE NA KRYTERIUM STOCHASTYCZNEJ INDYFERENCJI: PRZYPADEK POLSKI

Streszczenie

Artykuł przedstawia koncepcję *stochastycznych skal ekwiwalentności* (*SES*), która bazuje na kryterium stochastycznej indyferencji. SES jest dowolną funkcją, która transformuje rozkład wydatków określonej grupy gospodarstw domowych w taki sposób, że wynikowy rozkład jest stochastycznie indyferentny wobec rozkładu wydatków grupy gospodarstw odniesienia. Kryterium stochastycznej indyferencji jest także wykorzystane dla opracowania metody estymacji SES. Oszacowano nieparametryczne i parametryczne SES na podstawie Polskich Budżetów Gospodarstw Domowych za lata 2005–2010.

Słowa kluczowe: skale ekwiwalentności, stochastyczna indyferencja, estymacja, rozkład wydatków

EQUIVALENCE SCALES BASED ON STOCHASTIC INDIFFERENCE CRITERION:
THE CASE OF POLAND

Abstract

The paper presents the concept of the *stochastic equivalence scale* (*SES*), which is based on the stochastic indifference criterion. The *SES* is any function that transforms the expenditure distribution of a specific group of households in such a way that the resulting distribution is stochastically indifferent from the expenditure distribution of a reference group of households. The stochastic indifference criteria are also used in developing the method of the estimation of the *SES*. Non-parametric and parametric *SESs* are estimated using the Polish Household Budget Survey for the years 2005–2010.

Keywords: equivalence scale, stochastic indifference, estimation, expenditure distribution