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REALIZED VOLATILITY VERSUS GARCH AND STOCHASTIC VOLATILITY MODELS. THE EVIDENCE FROM THE WIG20 INDEX AND THE EUR/PLN FOREIGN EXCHANGE MARKET

1. INTRODUCTION

Volatility is an unobservable feature of each financial market. While we can directly observe the prices of instruments and their movement, we cannot observe their volatility. Volatility is a measure associated with risk and uncertainty connected with unexpected changes of an instrument price. This makes financial time series unpredictable. Since we cannot observe volatility, we only approximate it with the help of statistical models. The simplest and most naive measure of volatility of a given instrument is the standard deviation (or variance) of its price. However, the measure is static while the volatility of an instrument changes on day-to-day basis. That is why dynamic measures of volatility have gained more and more popularity.

Dynamic modelling of volatility can be divided into three groups [9]:

– taking advantage of volatility models such as the (Generalised) Autoregressive Conditional Heteroskedasticity models ((G)ARCH) and the Stochastic Volatility models (SV),

– implied volatility models, deriving volatility from the market option prices,

– realized volatility (RV), based upon the high-frequency data.

The aim of the article is to compare the estimates of volatility obtained from the parametric volatility models, namely the GARCH and the SV, with realized volatility RV as a non-parametric measure. We consider RV measures obtained from data of different frequencies and calculated within two approaches: with and without night returns. We take into account the Polish stock exchange market and the foreign exchange one: our sample consist of the WIG20 index and the EUR/PLN exchange rate and covers a hectic crisis period.

The remainder of the paper is organized as follows. Section 2 describes GARCH and SV models. The different types of calculating realized volatility are presented in Section 3. Section 4 presents the data and empirical results of estimation of the GARCH and the SV models. The comparison of realized volatility measures and in-sample estimated conditional volatility are shown in Section 5. Section 6 presents

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accuracy of forecasts obtained from the models in comparison to realized volatility. Section 7 concludes.

2. PARAMETRIC ESTIMATES OF VOLATILITY: GARCH AND SV MODELS

Although the financial time series are unpredictable, they display both temporal dependency in their second order moments and heavy-picked and tailed distributions. This phenomenon has been known since the work of Mandelbrot [17], but only after the introduction of the autoregressive conditionally heteroskedastic models the econometricians started modelling the dynamics of volatility.

The first model from the group of autoregressive models was the Autoregressive Conditional Heteroskedasticity (ARCH) model, proposed by Engle [11]. Let us define:

$$r_t = 100 \cdot \ln (S_t/S_{t-1}), \tag{1}$$

where r_t denotes the logarithmic return, and S_t the price of the instrument at time t.

Next, let:

$$y_t = r_t - E(r_t|\mathcal{F}_{t-1}), \tag{2}$$

where $E(r_t|\mathcal{F}_{t-1})$ denotes the expected conditional value of the instrument price (usually modelled with ARMA models). The assumption of the ARCH(q) model is that the time series $\{y_t\}$ is not correlated, but the variables are dependent and this dependence can be described with the help of a function whose arguments are the squared values of the lagged variables y_t:

$$y_t = \sigma_t \varepsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \ldots + \alpha_q y_{t-q}^2.$$

Where: $\{\varepsilon_t\} \sim iid(0,1)$ – independent random variables of the same distribution, $\alpha_0 > 0$, $\alpha_i \geq 0$, for $i > 0$ and σ_t^2 denotes the unobserved volatility (i.e. conditional variance).

A generalisation of the ARCH model was proposed by Bollerslev [4]. Again, it is assumed that the average of the logarithmic returns can be described with the help of the ARMA model. The GARCH(p, q) model takes the following form:

$$y_t = \sigma_t \varepsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

where: $\{\varepsilon_t\} \sim iid(0,1)$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$.

The advantage of the GARCH model over the ARCH is that the latter is often overparametrized, whereas in the practice the most common is plain vanilla GARCH(1,1).

Since the first GARCH(p, q) model was introduced, there have emerged lots of its modifications, which take into account many additional features of time series (such as leverage effect, long memory, different error distributions, see in: [5], [23]).

In many financial series the conditional variance estimated through GARCH(p, q) reveal strong persistence, characterized by:

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_i = 1 .$$

In this case we estimate IGARCH model, which can be written as:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} (1 - \alpha_i) \sigma_{t-j}^2 .$$

Under the IGARCH(1,1) the unconditional variance of y_t is not defined.

2.1. GARCH MODELS WITH NORMAL AND STUDENT DISTRIBUTIONS

For the standard normal distribution, and if we express the conditional mean equation as in Equation (2) and $y_t = \sigma_t \varepsilon_t$, the log-likelihood function is given by:

$$L_{norm} = -\frac{1}{2} \sum_{t=1}^{T} \left[\ln(2\pi) + \ln\left(\sigma_t^2\right) + \varepsilon_t^2 \right]$$

with T denoting the number of observation.

Due to the existence of fat tails in financial time-series, the use of a Student distribution in combination with GARCH models is suggested. In case of a Student distribution, the log-likelihood function is given by:

$$L_{Student} = T \left\{ \ln \Gamma\left(\frac{\upsilon + 1}{2}\right) - \ln \Gamma\left(\frac{\upsilon}{2}\right) - \frac{1}{2} \ln\left[\pi(\upsilon - 2)\right] \right\} - \frac{1}{2} \sum_{t=1}^{T} \left[\ln\left(\sigma_t^2\right) + (1 + \upsilon) \ln\left(1 + \frac{\varepsilon_t^2}{\upsilon - 2}\right) \right],$$

where υ denotes the degrees of freedom ($2 < \upsilon < \infty$) is estimated together with other parameters in the model and $\Gamma(\cdot)$ is the gamma function.

2.2. GARCH MODELS WITH LEVERAGE EFFECT

This effect was first described by Black [3] and it takes into account the fact that high negative returns are accompanied by higher volatility than the positive returns of the same magnitude. A volatility model commonly used in the case of leverage effect is the threshold GARCH, presented in Glosten, Jagannathan and Runkle [12]. The conditional variance equation takes the following form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p}(\alpha_i + \gamma_i N_{t-i})y_{t-i}^2 + \sum_{j=1}^{q}\beta_j \sigma_{t-j}^2,$$

where N_{t-i} is a dummy variable indicating the sign of innovations:

$$N_{t-i} = \begin{cases} 1 \; if \;\; y_{t-i} < 0 \\ 0 \; if \;\; y_{t-i} \geq 0 \end{cases}.$$

The parameter γ is interpreted as a leverage effect and it counts for any asymmetry on the volatility response to negative and positive shocks. If γ is positive, the past negative shocks have a larger impact on conditional variance than the positive shocks.

2.3. UNIVARIATE STOCHASTIC VOLATILITY MODELS

In the ARCH-type models volatility is made dependent on the variability of the past observations. It means, that volatility at the moment t is determined by the variables, the values of which are known up to time $t - 1$. An alternative approach was proposed by Clark [7] and Taylor [25]. In Stochastic Volatility models volatility is driven by additional, unobserved components (see e.g. [18]).

2.3.1. THE BASIC STOCHASTIC VOLATILITY MODEL

Let h_t^2 denote the latent volatility on the day t, while $\phi \in (-1;1)$ be a correlation coefficient. The basic SV model can be written in the following form:

$$y_t = \exp\left(\frac{1}{2}h_t\right)u_t,\; u_t \sim iid\; N(0,1)$$
$$h_t = \propto + \phi(h_{t-1} - \propto) + v_t,\; v_t \sim iid\; N(0,\tau^2),$$

where $h_0 \sim N(\propto, \tau^2)$.

The parameter ϕ is usually interpreted as so called volatility persistence parameter, while τ^2 is interpreted as *volatility of log-volatility* (see e.g. [15], [27]).

2.3.2. STOCHASTIC VOLATILITY MODEL WITH THE STUDENT DISTRIBUTION

The basic model can be changed into more sophisticated ones. One can incorporate into it the knowledge about the data characteristics, e.g. non-normality of distribution. In the SV model with the Student distribution it is assumed that the price of the instrument follows the Student distribution with v degrees of freedom:

$$y_t = \exp\left(\frac{1}{2}h_t\right)u_t,\ u_t \sim iid\ t_v,$$
$$h_t = \alpha + \phi(h_{t-1} - \alpha) + v_t,\ v_t \sim iid\ N(0, \tau^2).$$

2.3.3. STOCHASTIC VOLATILITY MODEL WITH LEVERAGE EFFECT

The model can be made even more sophisticated if one wishes to include the leverage effect into it.

The model takes the following form:

$$y_t = \exp\left(\frac{1}{2}h_t\right)u_t,\ t = 1, \ldots, N,$$
$$h_{t+1} = \alpha + \phi(h_t - \alpha) + \tau\, v_{t+1},\ t = 1, \ldots, N-1,$$
$$\begin{pmatrix} u_t \\ v_{t+1} \end{pmatrix} \sim iid.\ N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right].$$

Then, (h_t) and returns (y_t) have the following distributions:

$$h_{t+1} \,|\, h_t, \alpha, \phi, \tau^2 \sim N(\alpha + \phi(h_t - \alpha), \tau^2),$$
$$y_t \,|\, h_{t+1}, h_t, \alpha, \phi, \tau^2, \rho \sim N\left(\frac{\rho}{\tau}e^{h_t/2}(h_{t+1} - \alpha - \phi(h_t - \alpha)), e^{h_t}(1 - \rho^2)\right).$$

If $\rho < 0$, this means that the leverage effect exists and volatility tends to increase more when large negative returns are present in the market. The parameter ϕ denotes so called persistence of the volatility, which is said to be high if it the absolute value of this parameter is close to 1.

The model presented above is the one proposed in [13] and is an alternative to the model presented in [14]. The comparison of the models can be found in [26]. For more examples of univariate Stochastic Volatility Models and their implementations in WinBugs we refer the Reader to [27].

3. REALIZED VOLATILITY

Realized volatility is a non-parametric measure proposed by Andersen and Bollerslev [2] as a proxy for non-observable integrated volatility. Let us introduce the basic concept of integrated volatility (the whole section is based mainly on [16]).

We consider the following diffusion process for the logarithmic price $\ln S_t$:

$$d \ln S(t) = \propto(t)dt + \sigma(t)dW(t),\ t \geq 0,$$

where, as previously:
$S(t)$ – asset price at time t,
$d \ln S(t)$ – continuously compounded return,
$W(t)$ – standard Brownian motion process,
$\mu(t)$ – drift,
$\sigma(t)$ – volatility.

For very small time intervals, Δ, we obtain:

$$r(t, \Delta) \equiv S(t) - S(t - \Delta) \approx \propto (t - \Delta)\Delta + \sigma(t - \Delta)\Delta W(t),$$

where $\Delta W(t)$ is normally distributed with zero mean and standard deviation equal to Δ.

When daily one-period return is considered:

$$r_t \equiv S(t) - S(t-1) = \int_{t-1}^{t} \propto(s)ds + \int_{t-1}^{t} \sigma(s)dW(s).$$

Additionally, conditionally on the sample path of the drift and the spot volatility,

$$r_t \sim N\left(\int_{t-1}^{t} \propto(s)ds,\ \int_{t-1}^{t} \sigma^2(s)ds\right),$$

where $\int_{t-1}^{t} \sigma^2(s)ds$ is so-called integrated volatility (integrated variance).

However, integrated volatility is not observed in practice. Realized volatility provides a consistent non-parametric estimate of a financial instrument prices variability over a given time interval and therefore is often treated as a proxy for integrated volatility.

The realized volatility is denoted $RV_t(\Delta)$ and defined as:

$$RV_t(\Delta) \equiv \sum_{i=1}^{N} r_{t,i}^2,$$

where:
Δ – is a number of equally spaced intervals within a day,
$r_{t,i}$ – is a logarithmic return on day t at time interval i; with $i = 1, \ldots, N$ and $t = 1, \ldots, T$.

In the present work realized volatility is calculated in two ways both with and without night return (RV1 and RV2 respectively).

As a non-parametric and easy to obtain estimate, realized volatility is very commonly used in assessing the quality of conditional variance forecasts from the GARCH and the SV models within Mincer-Zarnowitz regression [19], which is in the form of ex-post volatility regression:

$$\breve{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\sigma}_t^2 + a_t,$$

where:
$\breve{\sigma}_t^2$ is ex-post volatility (e.g. realized volatility) at time t,
$\hat{\sigma}_t^2$ is estimated (in-sample) or forecasted (out-of sample) volatility at time t,
a_t – independent and identically distributed; $a_i \sim N(0,1)$.
α_0 and α_1 are parameters to be estimated.

If the model for conditional variance is well specified, and if $E\left(\breve{\sigma}_t^2\right) = \sigma_t^2$, we should have: $\alpha_0 = 0$, $\alpha_1 = 1$ [2]. According to specific features of financial data series the value of R^2 is usually low (even less than 5%).

Our approach is to use realized volatility for two purposes. First, we check if realized volatility is close to in-sample parametric estimates of volatility from GARCH and SV models. Having in-sample estimates we are able to compare two different measures of realized volatility, one of which includes night return and the other does not. In this case $\breve{\sigma}_t^2$ is the realized volatility and $\hat{\sigma}_t^2$ is the estimated conditional variance from the GARCH or the SV models.

Second, we use GARCH and SV models forecasts to examine which realized volatility, including night return or not, from lower or higher frequencies, is closer to forecasts obtained from these models. In this case $\hat{\sigma}_t^2$ stands for the forecast of conditional variance from the GARCH or the SV models.

4. DATA DESCRIPTION AND EMPIRICAL RESULTS

4.1. DATA

Our data set consists of daily and intradaily prices of the WIG20 index and the EUR/PLN exchange rate over the period 2006-01-02 to 2009-11-06. We transform them into percentage logarithmic returns. The data are taken from www.stooq.pl. We estimate conditional variance from both the GARCH model (starting at 2006-01-10) and the SV model (starting at 2008-01-03) and introduce them into the Mincer-Zarnowitz in-sample regression together with the realized volatility from 2008-01-02 to 2009-07-31. Finally, Mincer-Zarnowitz regression is performed on the forecasts of the conditional variance from the GARCH and the SV models – there are 70 forecast from 2009-08-01 to 2009-11-06.

All calculations were done using OxMetrics 6.0 [10] and WinBugs 1.4. In case of WinBugs we took advantage of the models written by Yu [26, 27] and available at his webpage.

4.2. ESTIMATION OF GARCH MODELS

The GARCH model was chosen by taking into account information criteria. Four models were compared: the most popular GARCH(1,1) with the normal and the Student distributions and the GJR-GARCH model [12] with normal and Student distributions. Here we present values of the Schwarz Information Criterion. Other considered criteria (Akaike and Hannan-Quinn) give the same indication.

Table 1

Comparison of GARCH models estimated for the WIG20 index

Model	GARCH(1,1)	GARCH(1,1)	GJR-GARCH(1,1)	GJR-GARCH(1,1)
distribution	normal	Student-*t*	normal	Student-*t*
SIC	4,075	4,071	4,067	4,065

The GJR model with the Student distribution occurred to be the best. In Table 2 the parameters estimated for GARCH(1,1) and GJR-GARCH(1,1) (both with the Student distribution) are presented.

Table 2

Estimates for GARCH(1,1) and GJR-GARCH(1,1) for returns of the WIG20 index

	α_0	α_1	β_1	γ	v	LL
GARCH(1,1)	0,050	0,054	0,933		11,037	–1789,23
	(0,025)	(0,014)	(0,016)			
GJR-GARCH(1,1)	0,051	0,005	0,945	0,071	12,316	–1796,60
	(0,026)	(0,011)	(0,019)	(0,023)		

For the GARCH(1,1) we obtain statistically significant estimates of the parameters. In the GJR-GARCH(1,1) model we obtain statistically significant parameter of the leverage effect and statistically insignificant parameter of the past squared shocks. Introducing leverage effect into conditional variance equation also resulted in increasing the degrees of freedom in the Student distribution.

Table 3

Comparison of GARCH models estimated for the EUR/PLN exchange rate

Model	GARCH(1,1)	GARCH(1,1)	IGARCH(1,1)	IGARCH(1,1)
distribution	normal	Student-*t*	normal	Student-*t*
SIC	1,639	1,622	1,632	1,615

In the case of the EUR/PLN exchange rate the IGARCH model with the Student distribution occurred to be the best. The estimates are presented in Table 4 together with the standard GARCH(1,1) model.

Table 4

Estimates of GARCH(1,1) and IGARCH(1,1) for the EUR/PLN exchange rate

	α_0	α_1	β_1	v	LL
GARCH(1,1)	0,02	0,08	0,92	7,75	–709,67
	(0,01)	(0,02)	(0,02)		
IGARCH(1,1)	0,02	0,08	0,92	7,51	–709,73
	(0,01)	(0,02)			

4.3 ESTIMATION OF STOCHASTIC VOLATILITY MODELS

As an alternative to GARCH we estimated the Stochastic Volatility model. We compared three models: the basic Stochastic Volatility model, the model with leverage effect and the model with the Student distribution.

Since in case of SV models, distribution of the time series at moment t is unknown conditional upon the observations up to the moment $t-1$, estimation of the models' parameters with the help of the standard methods based upon the maximum-likelihood is not possible. Broto and Ruiz [6] provide a survey of methods of the SV models estimations, while Kim et al. analyze the efficiency of their estimation [15]. From the various methods mentioned in the papers we chose Bayesian method based upon the MCMC (Markov Chain Monte Carlo) algorithm [27]. An example of the application of the Bayesian methodology to the estimation of stochastic volatility processes in analysis of the financial time series can be found e.g. in: [8], [21] and [22].

For each of the series we estimated three mentioned models and compared them with the help of DIC (Deviance Information Criterion). The criterion was first introduced by Spiegelhalter [24] as a measure of model comparison and adequacy. It is given by the following expression:

$$DIC(m) = 2\overline{D(\theta_m, m)} - D\left(\overline{\theta_m}, m\right) = 2D\left(\overline{\theta_m}, m\right) + 2p_m,$$

where:

$$D\left(\overline{\theta_m}, m\right) = -2\log f\left(y \mid \theta_m, m\right)$$

and $\overline{D(\theta_m, m)}$ is the posterior mean of the deviance measure $D\left(\overline{\theta_m}, m\right)$. The value p_m can be interpreted as the number of "effective" parameters of the model m: $p_m = \overline{D(\theta_m, m)} - D\left(\overline{\theta_m}, m\right)$. Eventually, $\overline{\theta_m}$ is the posterior mean of parameters involved in the model m [20]. The criterion is considered as a generalization of the Akaike Information Criterion (AIC) [1]. The minimum value of the criterion indicates the model which offers the best short-term predictions. Since we are interested especially in the prediction power of the models, we report here the values of DIC and use the criterion to select the best model. For the Reader's information we report also the values of the Bayes Information Criterion (BIC) which is based upon the Schwarz criterion [9]. The BIC for the model m is defined as:

$$BIC(m) = D\left(\hat{\theta}_m, m\right) + d_m \log(n),$$

where $\hat{\theta}_m$ are the maximum likelihood estimates of parameters θ_m of model m and d_m is the dimension of θ_m. As previously, $D(\hat{\theta}_m, m)$ is the deviance measure of model m.

The tables below present the results of models comparison:

Table 5

Comparison of SV models estimated for the WIG20 index

Model	basic SV	*t*-SV	leverage SV
BIC	1768.108	1777.850	1465.22
DIC	1756.880	1760.980	1678.030

Table 6

Comparison of SV models estimated for the EUR/PLN exchange rate

Model	basic SV	*t*-SV	leverage SV
BIC	916.610	959.512	755.830
DIC	913.312	982.680	860.019

What is interesting, in both cases the basic SV model performed better than the model with the Student distribution. DIC criterion reached its minimum for the models with the leverage effect. Thus, we further present the results only from this model.

Since the estimation was performed using the WinBUGS software for Bayesian modeling, we treat each parameter of the model as a random variable and present the obtained mean, median, standard deviation and 95% confidence interval for each of them. The MC error reported in each table denotes the so called Monte Carlo error which measures the variability of each estimate due to the simulation (see e.g.: [20]). We also present the obtained posterior densities of each parameter. In the case of stochastic volatility we present the 95% confidence interval of the estimated values.

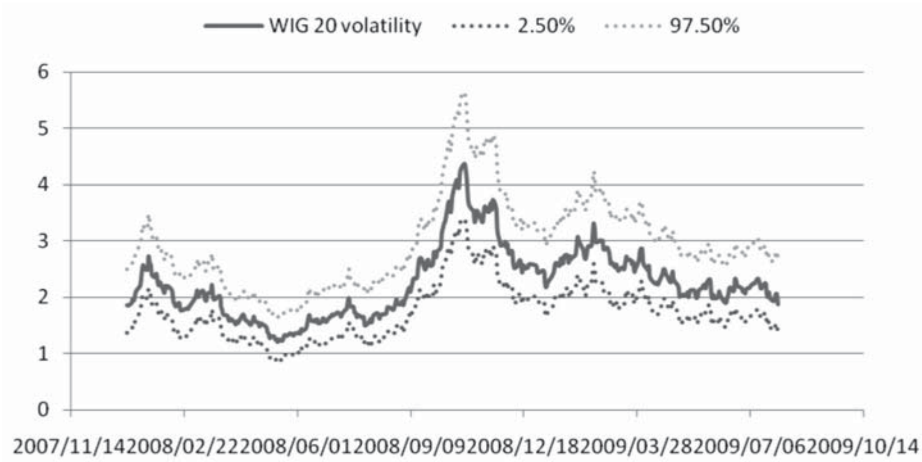

Figure 1. 95% confidence intervals obtained for volatility of WIG20 (leverage SV model)

Table 7

Descriptive statistics and 95% confidence interval for parameters of the WIG20 index volatility

Parameter	Mean	Standard deviation	MC error	2.5%	Median	97.5%
τ	0.132	0.025	0.002	0.089	0.129	0.196
ρ	–0.582	0.154	0.012	–0.814	–0.606	–0.193
ϕ	0.982	0.010	0.001	0.956	0.983	0.996
α	0.024	0.016	0.001	0.003	0.021	0.065
$\propto$	1.252	0.301	0.020	0.612	1.272	1.806

To estimate the parameters of the model we took advantage of the programs written by Yu [26, 27]. In case of the three SV models the parameters: τ, ϕ and $\propto$ have the following prior distributions: τ – the inverse Gamma, $\propto$ – the Normal, while $\phi^* = (\phi + 1)/2$ – the Beta distribution. The additional parameter ρ, present in the leverage SV model is uniformly distributed: $\rho \sim U(-1, 1)$.

Based upon the results displayed in Table 7 we can state that the leverage effect is present in the volatility of the WIG20 index. The parameter ρ is negative and both ends of the 95% confidence interval are negative. At the same time we can observe that the value of the parameter ϕ is close to 1, which is a sign of a high persistence of volatility. Figures 2 to 6 present the obtained posterior densities of each parameter.

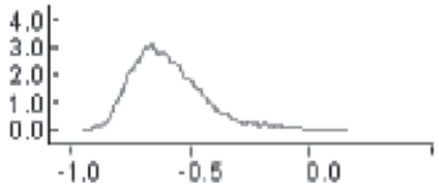

Figure 2. Density of parameter ρ

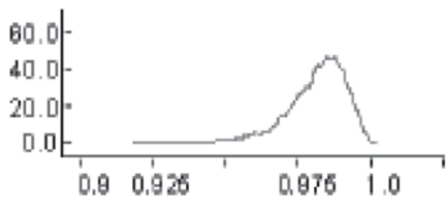

Figure 3. Density of parameter ϕ

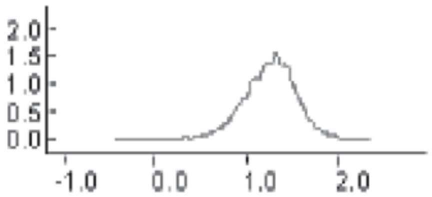

Figure 4. Density of parameter μ

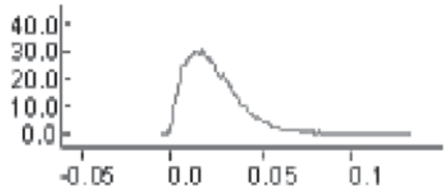

Figure 5. Density of parameter α

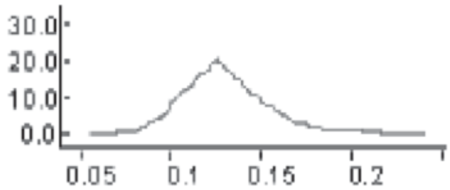

Figure 6. Density of parameter τ

Figure 7 presents the 95% confidence interval obtained for the volatility of the EUR/PLN. Similarly to the case of the WIG20, there is a pick at the end of 2008 and at the beginning of 2009. However, in the case of the WIG20, where the second pick is smaller, in the case of the EUR/PLN both of them reach approximately the same value.

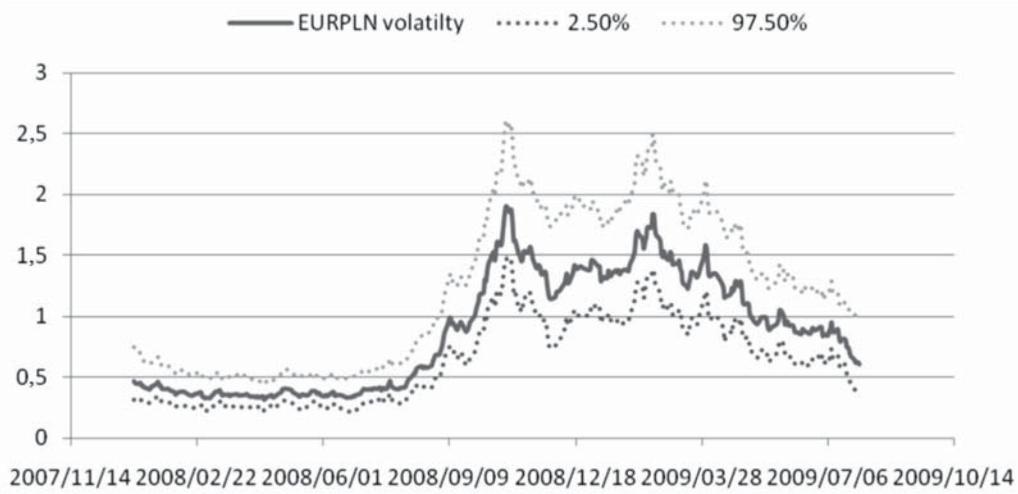

Figure 7. 95% confidence intervals obtained for volatility of the EUR/PLN exchange rat (leverage SV model)

In the case of the EUR/PLN exchange rate we do not observe the leverage effect, since the posterior mean of % takes positive value of 0.425. Also, if we take a look at its estimated density (Figure 8), we can observe that it is very unlikely that it will take a negative value. However, the value taken by the parameter ϕ indicates again the high persistence of volatility. Figures 8 to 12 present the densities obtained for each parameter, while their descriptive statistics are presented in Table 8.

Table 8

Descriptive statistics and 95% confidence interval for parameters of the EUR/PLN foreign exchange volatility

Parameter	Mean	Standard deviation	MC error	2.5%	Median	97.5%
τ	0.162	0.041	0.004	0.099	0.156	0.257
ρ	0.425	0.187	0.015	0.016	0.434	0.749
ϕ	0.988	0.008	0.001	0.968	0.990	0.999
α	–0.014	0.008	0.000	–0.033	–0.013	–0.002
$\propto$	–1.381	0.487	0.035	–2.331	–1.376	–0.467

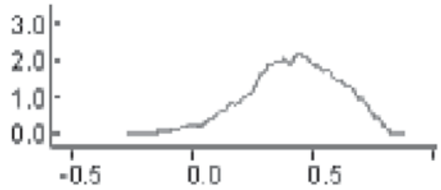

Figure 8. Density of parameter ρ

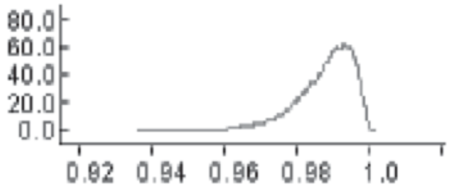

Figure 9. Density of parameter ϕ

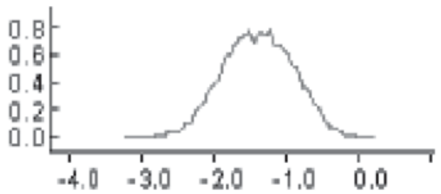

Figure 10. Density of parameter μ

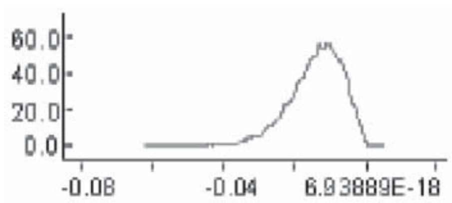

Figure 11. Density of parameter α

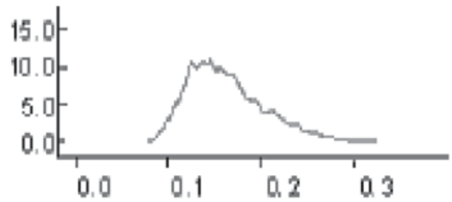

Figure 12. Density of parameter τ

5. REALIZED VOLATILITY MEASURES AND CONDITIONAL VARIANCES FROM GARCH AND SV MODELS

5.1. REALIZED VOLATILITY IN DIFFERENT FREQUENCIES

In Figure 13 two kinds of realized volatility (with and without night return) and from different frequencies are presented, together with the estimated conditional variance from the GJR-GARCH and the SV models.

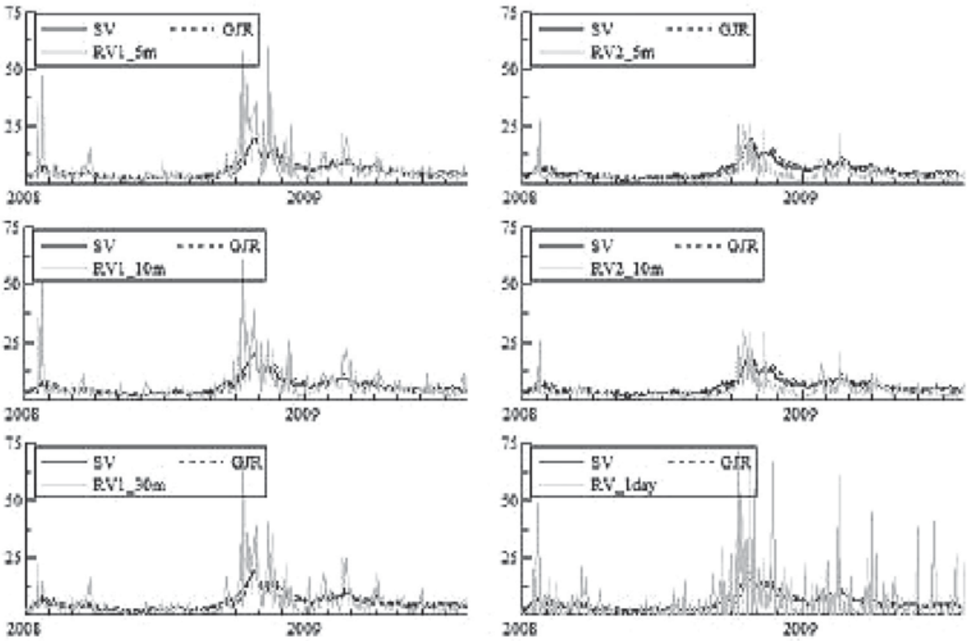

Figure 13. Realized volatility based on 5-, 10- and 30-minute returns with (RV1) and without (RV2) night return and square of daily returns (RV_1day) compared to estimates of conditional variance from the GJR-GARCH(1,1) and the SV models.

In Figure 13 realized volatility with night return (RV1) takes significantly higher values than realized volatility without night return (RV2). In other words, the realized volatility without night returns is much more smoothed. Such spectacular differences are not recognizable, however, if we compare realized volatility with night returns but from different frequencies – based on 5- and 10-minute returns (e.g. RV1_5m and RV1_10m). Eventually, if one would like to compare conditional variance from the GJR-GARCH and the SV models, the estimates of volatility are quite close to each other and the direction of changes (increase or decrease) in volatility is similar to that obtained within realized volatility.

5.2. COMPARISON OF REALIZED AND CONDITIONAL VARIANCE IN-SAMPLE (MINCER-ZARNOWITZ REGRESSION)

We compare the realized volatility measures among themselves. Because we do not know which approach, parametric or nonparametric, gives us the best estimates of unobserved volatility, we use the Mincer-Zarnowitz regression to our estimates. If the estimated parameters in this regression are as expected, it means α_0 close to 0, α_1 close to 1 and R^2 is not lower than 30%, this will indicate that the differences between the compared estimates are not huge and if so, both of them are good estimates of variance.

Table 9

Mincer-Zarnowitz in-sample regression for the WIG20 volatility

$$RV_t = \alpha_0 + \alpha_1 \hat{\sigma}_t^2 + u_t$$

	RV1			RV2			RV		
	α_0	α_1	R^2	α_0	α_1	R^2	α_0	α_1	R^2
	5 minutes			**5 minutes**			**1 day**		
GJR-GARCH(1,1)	–0,817	1,267	0,342	0,010	0,655	0,360	0,857	0,899	0,094
	(0,573)	(0,089)		(0,285)	(0,044)		(0,907)	(0,140)	
SV	–2,246	1,503	0,425	–0,602	0,771	0,440	–1,351	1,282	0,170
	(0,567)	(0,088)		(0,282)	(0,044)		(0,920)	(0,143)	
	10 minutes			**10 minutes**			**30 minutes**		
GJR-GARCH(1,1)	–0,969	1,301	0,323	–0,174	0,693	0,356	–0,953	0,527	0,350
	(0,613)	(0,095)		(0,304)	(0,047)		(1,188)	(0,0816)	
SV	–0,425	1,177	0,591	–0,218	0,838	0,723	–2,418	1,433	0,449
	(0,179)	(0,033)		(0,095)	(0,017)		(0,514)	(0,078)	

RV1 stands for realized volatility calculated with night returns and RV2 is calculated without night returns. 5, 10, and 30 minutes denote the frequencies of returns. RV 1 day stands for square of daily returns.

Parameters of the Mincer-Zarnowitz regression presented in Table 9 imply that the realized volatility is a better measure of unobserved volatility than the commonly used squares of returns (R^2 values are the lowest). When we compare realized volatility from different frequencies, in both cases: RV1 and RV2, estimates of conditional variance form the GJR-GARCH models are closer to realized volatility based on 5-minute returns, while estimates of conditional variance from the SV models are significantly closer to realized volatility based on 10-minute returns (and not 30-minute returns). When we compare two types of realized volatility, with (RV1) and without night return (RV2), the realized volatility without night return seems to perform better, especially for RV measure based on 10-minute returns.

In Table 10 the results of the Mincer-Zarnowitz in-sample regression for the EUR/PLN exchange rate are presented.

Table 10

Mincer-Zarnowitz regression in-sample for the EUR/PLN exchange rate volatility

	RV1			RV2			RV		
	α_0	α_1	R^2	α_0	α_1	R^2	α_0	α_1	R^2
	10 minutes			**10 minutes**			**1 day**		
IGARCH	0,120	1,083	0,453	0,061	0,948	0,508	0,163	0,858	0,148
	(0,084)	(0,060)		(0,065)	(0,046)		(0,145)	(0,103)	
SV	–0,102	1,347	0,618	–0,105	1,151	0,661	–0,161	1,222	0,2657
	(0,071)	(0,053)		(0,056)	(0,042)		(0,1369)	(0,102)	
	20 minutes			**20 minutes**			**30 minutes**		
IGARCH	0,104	1,099	0,451	0,085	1,056	0,454	0,105	1,125	0,448
	(0,086)	(0,061)		(0,082)	(0,059)		(0,088)	(0,062)	
SV	–0,113	1,357	0,606	–0,119	1,303	0,606	–0,125	1,399	0,612
	(0,074)	(0,055)		(0,071)	(0,053)		(0,075)	(0,056)	

The realized volatility of any frequency in intraday data is a better estimate of volatility than squares of daily returns (R^2 value is the lowest for the latter one). Similar to the WIG20 case, estimates of conditional variance from stochastic volatility models are closer to realized volatility than that from IGARCH model. When comparing realized volatility with and without night return, the latter is slightly closer to conditional variance estimates from parametric models. Moreover, in the case of the model for foreign exchange EUR/PLN the parameters estimated in the Mincer-Zarnowitz regression are as expected: α_0 is not statistically significantly different from zero and α_1 is very close to one.

6. THE ACCURACY OF GARCH AND SV FORECASTS – COMPARISON TO REALIZED VOLATILITY MEASURES

As a next step we computed 70 one-step ahead forecasts of the conditional variance within the GJR-GARCH (WIG20), the IGARCH (EUR/PLN) and the SV models (WIG20 and EUR/PLN). Next we compare them to different measures of realized volatility. In this section we answer the question: which frequency or approach of calculating realized volatility gives the non-parametric estimates of variance which are the closest to the variance forecast from the GARCH and the SV models. We take into account the following forecast error measures:

$$MAE = \frac{1}{T}\sum_{t=1}^{T}\left|\sigma_t^2 - RV_t\right|,$$

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\sigma_t^2 - RV_t\right)^2},$$

$$MAPE = \frac{1}{T}\sum_{t=1}^{T}\left|\frac{\sigma_t^2 - RV_t}{\sigma_t^2}\right|,$$

$$AMAPE = \frac{1}{T}\sum_{t=1}^{T}\left|\frac{\sigma_t^2 - RV_t}{\sigma_t^2 + RV_t}\right|.$$

Forecast error measures from Mincer-Zarnowitz out-of-sample regression for WIG20 volatility are presented in Table 11.

Table 11

Forecasts error measures for differently calculated realized volatility of the WIG20 index

	MAE		RMSE		MAPE		AMAPE	
	GJR-GARCH (1,1)	SV	GJR-GARCH (1,1)	SV	GJR-GARCH (1,1)	SV	GJR-GARCH (1,1)	SV
RV1_5m	1.542	1.321	2.562	2.358	0.450	0.376	0.180	0.152
RV2_5m	1.355	1.152	1.602	1.326	0.376	0.336	0.239	0.215
RV1_10m	1.575	1.352	2.308	2.089	0.459	0.386	0.194	0.166
RV2_10m	1.531	1.347	1.793	1.521	0.429	0.393	0.277	0.257
RV1_30m	1.813	1.602	2.543	2.312	0.526	0.460	0.256	0.231
RV_1day	3.879	3.795	5.509	5.442	1.118	1.095	0.535	0.530

MAE stands for Mean Absolute Error, RMSE – Root Mean Squared Error, MAPE – Mean Absolute Percentage Error, AMAPE – Adjusted Mean Absolute Percentage Error.

We compare conditional variance forecasts to realized volatility calculated in different frequency (5, 10 and 30 minutes) and in a different way (with or without night returns). Generally, we conclude that the realized volatility from the highest examined frequency performs better than from the lower. It is in agreement with the results of Doman [10], where realized volatility from the highest frequency performs the best when forecast error measures are taken into account. However, in our study when comparing 5-minute realized volatility calculated with and without night return we do not have clear answer which of these two measures of realized volatility is better.

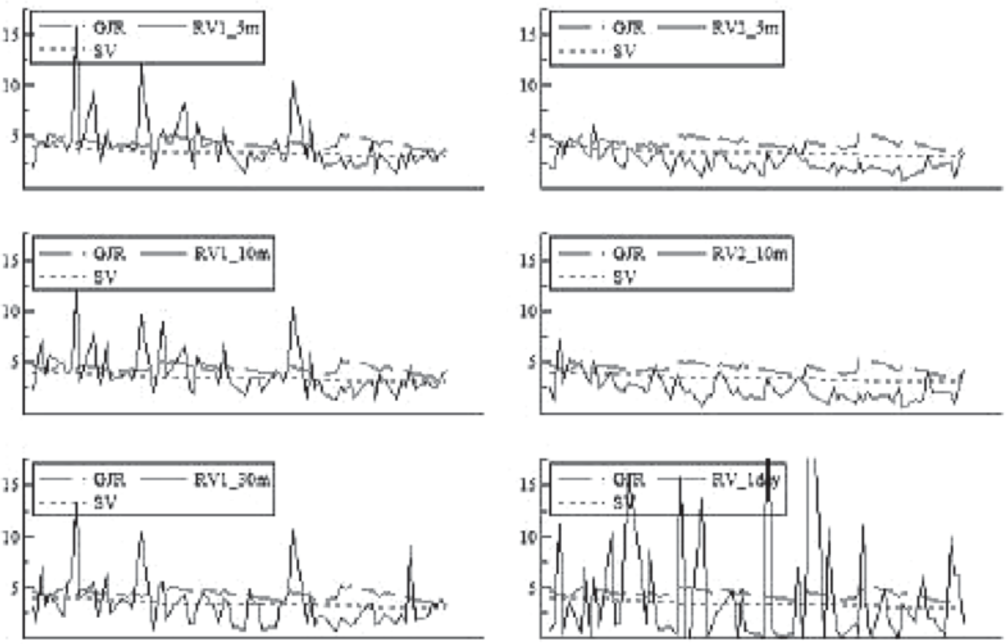

Figure 14. Conditional variance forecasts from GARCH and SV models with realized volatility of the WIG20 index

In Figure 14 we show conditional variance forecasts from the GJR-GARCH and SV models with realized volatility from different frequencies. The realized volatility with night return is generally higher than variance forecast, while realized volatility without night return is generally lower. When we look at the squared returns (realized volatility based on 1-day return), this measure is definitely the least adequate.

Table 12

Forecasts error measures for differently calculated realized volatility of the EUR/PLN exchange rate

	MAE		RMSE		MAPE		AMAPE	
	IGARCH(1,1)	SV	IGARCH(1,1)	SV	IGARCH(1,1)	SV	IGARCH(1,1)	SV
RV1_5m	0.253	0.281	0.325	0.381	0.435	0.810	0.206	0.254
RV2_5m	0.227	0.212	0.289	0.298	0.377	0.611	0.204	0.213
RV1_10m	0.284	0.299	0.379	0.437	0.491	0.864	0.229	0.259
RV2_10m	0.280	0.275	0.371	0.413	0.477	0.793	0.233	0.245
RV1_30m	0.739	0.922	1.136	1.292	1.305	2.630	0.335	0.488
RV_1day	0.464	0.416	0.625	0.634	0.800	1.191	0.479	0.491

We calculate forecast error measures having 70 one-step ahead forecasts from IGARCH model for the EUR/PLN exchange rate. The error measures presented in Table 12 show that the lowest values of the errors are – again – obtained for the highest frequency, which is 10-minutes in this case. The conditional variance from the IGARCH and SV models together with differently calculated realized volatility are presented in Figure 15.

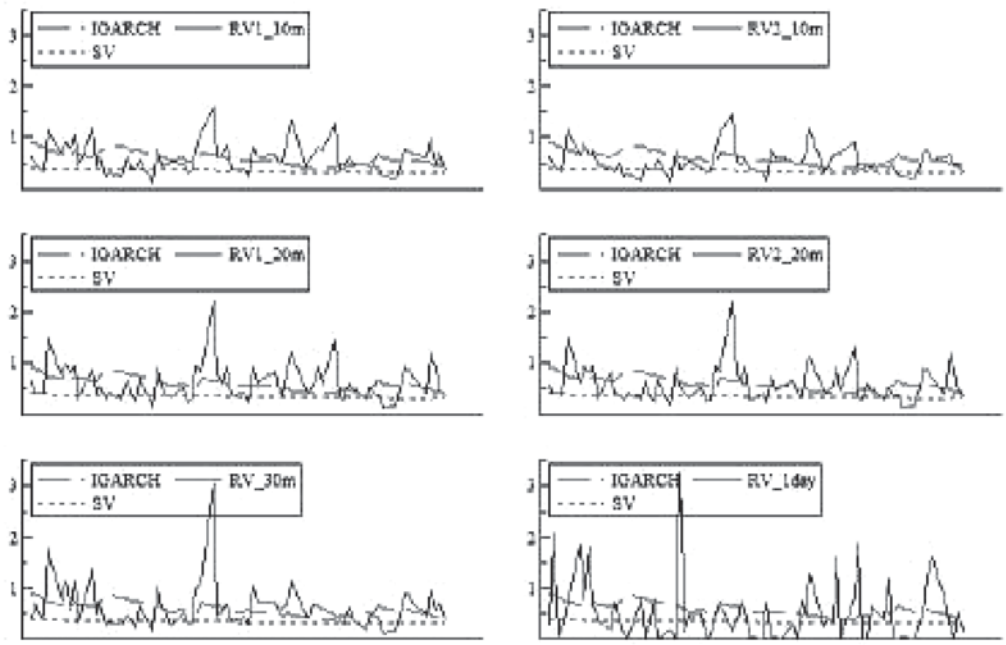

Figure 15. Conditional variance forecasts and realized volatility of the EUR/PLN exchange rate

7. CONCLUSIONS

The aim of the paper was to compare three types of volatility estimates: GARCH, SV and RV for the given instruments on the Polish financial markets: the stock exchange and the foreign currency ones. Based upon our samples: the WIG20 index and the EUR/PLN foreign exchange rate, both covering the crisis period, we computed the mentioned measures of volatility and compared them using the simplest and most commonly applied technique: the Minzer-Zarnowitz regression. Contrary to most commonly applied procedure, we used not only the estimates of RV to compare SV and GARCH models, but we also compared the RV measures among themselves. Since volatility is not observable, we cannot decide estimates of which models are the best measures of volatility of a financial instrument. Thus, we assume that two measures of volatility are equally good if the R^2 coefficient of the regression is quite high, the parameter α_0 is close to 0 and α_1 to 1. This assumption allowed us to compare the RV measures obtained from the data of different frequencies.

In the case of the WIG20 index and the EUR/PLN exchange rate it emerged that the square of daily returns is the worst estimate of volatility from all applied RV models. In both cases RV computed based upon 10-minute returns without the night return performed the best (according to R^2 measure). Moreover, in the case of the WIG20 index the conditional variance from GJR-GARCH model was closer to the RV computed based upon 5-minute returns, while the Leverage-SV model – to the one computed based upon the 10-minute returns. However, in case of the EUR/PLN exchange rate both the SV and the IGARCH models were close to the RV computed based upon 10-minute returns.

Eventually, we compared the RV estimates with conditional variance from the GARCH and the SV models estimated for the WIG20 index and the EUR/PLN foreign exchange. In general, it seems that RV computed from data of higher frequency performed better than others. The same conclusion can be drawn in the case of the EUR/PLN exchange rate, although here the preferred model was the one without the night return (in the case of the WIG20 index the results were ambiguous).

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ZMIENNOŚĆ ZREALIZOWANA WOBEC MODELI GARCH I MODELI ZMIENNOŚCI STOCHASTYCZNEJ NA POLSKIM RYNKU KAPITAŁOWYM

Streszczenie

Celem artykułu jest porównanie oszacowań zmienności uzyskanych z modeli parametrycznych: GARCH i SV z oszacowaniem uzyskanym na podstawie zmienności zrealizowanej szacowanej w oparciu o dane różnej częstotliwości. W badaniu wzięto pod uwagę zwroty z wybranych instrumentów polskiego rynku finansowego: indeks WIG 20 oraz kurs walutowy EUR/PLN. Ujęta w badaniu próba objęła okres kryzysu finansowego, co stanowi istotne uzupełnienie wyników prezentowanych do tej pory w literaturze.

Słowa kluczowe: zmienność zrealizowana, SV, GARCH, prognozowanie zmienności

REALIZED VOLATILITY VERSUS GARCH AND STOCHASTIC VOLATILITY MODELS.
THE EVIDENCE FROM THE WIG20 INDEX AND THE EUR/PLN FOREIGN EXCHANGE MARKET

Summary

The aim of the article is to compare the estimates of the volatility obtained from the parametric models: the GARCH and the SV with the estimates based upon the Realized Volatility approach, whereas the estimates from the RV are obtained from the data of different frequencies. The data sample consists of the WIG20 index and the EUR/PLN exchange rate and covers the hectic crisis period. Hence, the presented results can be viewed as an extension of the results of the studies presented up to date.

Key words: realizded volatility, SV, GARCH, volatility forecasting