PRZEGLĄD STATYSTYCZNY
R. LVI – ZESZYT 3-4 – 2009

HELENA GASPARS-WIELOCH

THE DISCRETE LOCATION PROBLEM FOR A CHAIN OF HOMOGENEOUS FACILITIES

1. INTRODUCTION

Location problems arouse interest of many people, but everyone formulates assumptions in a different way. In some papers a continuous approach is presented [2], [11], in others – a discrete one [5]. Some authors are considering the possibility to locate facilities in a linear market, e.g. along the beach [5], [11], others concentrate on bi-dimensional problems, but simultaneously demonstrate that they can be reducible to a one-dimensional case [3]. Objectives are also diverse. The goal of the research may consist, for instance, in maximizing the difference between the demand captured and the demand lost [13], in minimizing the distance between the facility and the highest number of potential clients [11] or in minimizing the total cost both for building and operating facilities as well as for servicing the customer demands [2]. The cases discussed in the literature concern both the location of facilities belonging to one proprietor [4] as well as the location of competitive facilities [1], [5]. In some articles the location is established for a set of similar facilities [5], [10], in others – for a complex of different facilities [14]. In some models location decisions are only considered [5], while other models incorporate additional decisions concerning, e.g. price and production levels [4]. We can find examples where facilities capacity, which may be treated as limited or not, is recognized as a decisive factor influencing the optimal solution [1], [2].

The main objective of this paper is to formulate models determining the best location strategy for a chain of homogeneous facilities, which belong to a single owner. Therefore, the model's goal is to maximize his or her total profit and not the profit of each facility separately. In the research facilities are defined as goods or services distribution centers (e.g. shops, restaurants, petrol stations, post offices, repair services) and are represented as points, not lines (e.g. railways) or polygones. Services are not provided virtually. The facilities' homogeneity means that they offer the same products at the same price, with the same quality of services and with similar opening hours. Although there are lots of contributions devoted to a continuous approach [8], [12], many researchers stress that the analysis of a region as a homogeneous space without geographical barriers is not appropriate [7]. That is why formulated models should be applicable to a discrete approach. It usually means that the location for p facilities is chosen from a set of n possible places (where $p < n$). Here, each potential facility is assigned to a particular location which has already been investigated from an administrative and transport point of view. Therefore, the decision-maker does not care about the place where to build a facility, but which facilities should be built. In other words, the goal is to choose, from a set of n potential facilities, that subset of p facilities which allows the objective function to achieve the most profitable value. Sections 2-4 describe three possible models designed to establish the best location strategy in view of assumptions given above. Section 5 presents suggestions how to deal with research limitations. Section 6 presents a conclusion.

2. FIRST APPROACH – LOCATION PROBLEM VERSUS RESOURCES ALLOCATION PROBLEM

The comparison of two different issues – the resources allocation problem and the location problem – reveals their significant similarities. In the first case the target is to distribute scarce resources among alternative activities. In the second one, the objective is to assign a limited number of facilities (in view of a given budgetary constraint) to different delimited areas (e.g. quarters, cities, regions). Although this analogy occurs, we can not recognize that the facilities considered in location issues are as totally homogeneous as the resources are in resources allocation problems, because even homogeneous facilities in terms of products, services and prices, differ from each other in their location! That is why, in contrast to resources allocation, in locations problems not only the number of facilities activated at each area has to be defined, but also the combination, which entails a significant growth of the number of possible solutions. Consider n regions and m_j potential facilities in region j. If we want to know how many facilities should be built in n regions we choose one of

$$\prod_{j=1}^{n}\left(m_j + 1\right) \tag{1}$$

possible strategies, but if we also consider the combination, the number of variants rises to

$$2^{\left(\sum_{j=1}^{n} m_j\right)} \tag{2}$$

Additionally, the location factor influences the number of combinations for which the profit has to be estimated. When regions are far from each other and profits generated by a facility depend only on other facilities built at the same area, the number is equal to

$$\sum_{j=1}^{n} 2^{m_j}. \tag{3}$$

However, when regions are close to each other, the facility activation may influence profits generated by facilities opened both in the same area as well as in adjacent areas. Then, the profit estimation is required for more cases:

$$2^{\left(\prod_{j=1}^{n} m_j\right)}. \qquad (4)$$

If in the optimization tasks concerning resources allocation, the total quantity of resources to allocate is not set in advance, the problem may be solved with the aid of the dynamic programming or some simplified procedures such as the local extrema method (for each activity the quantity connected with the highest profit is chosen) or the marginal profits method (resources are allocated as long as marginal profits are non-negative). The last method finds applications only in non-increasing marginal profits functions' cases. Methods mentioned above, except for local extrema method, are also used when the quantity to allocate is known in advance (then, using the marginal profits procedure, facilities built in the first order are those connected with the highest marginal profit). Algorithms characteristic of resources allocation issue may be applied to facilities location on condition that for p_j facilities activated in region j the combination connected with the highest profit is the only one to be taken into account. Moreover, these methods may be used only if interdependency between facilities built in different regions does not occur.

Table 1

Estimated cumulative profits for each region

p_j	Region A		Region B		Region C	
	set of facilities	profit	set of facilities	profit	set of facilities	profit
1	A1	3	**B1**	**3,5**	**C1**	**3**
	A2	**4**	–	–	C2	2
	–	–	–	–	**C3**	**3**
2	**A1, A2**	**5**	–	–	C1, C2	4
	–	–	–	–	**C1, C3**	**6**
	–	–	–	–	C2, C3	3,5
3	–	–	–	–	**C1, C2, C3**	**5,5**

Source: own calculations.

Table 1 presents cumulative profits estimated for regions A, B, C on the assumption that $m_1 = 2$, $m_2 = 1$, $m_3 = 3$. Symbols A1, A2, B1, C1, C2, C3 design potential facilities. Profits depend on the number and set of facilities activated. Cumulative profits are estimated for all possible combinations and the best ones are in bold.

Marginal profits are given in table 2. They are calculated according to the following formula:

$$c'_j(p_j) = c_j(p_j) - c_j(p_j - 1) \qquad p_j = 1, 2, \ldots, m_j \qquad (5)$$

where $c_j(p_j)$ signifies the estimated profit for the best combination of p_j facilities activated in region j.

Table 2

Marginal profits calculated for the best variants

p_j	Region A		Region B		Region C	
	set of facilities	profit	set of facilities	profit	set of facilities	profit
1	–	–	**B1**	**3,5**	**C1**	**3**
	A2	**4**	–	–	–	–
	–	–	–	–	**C3**	**3**
2	**A1, A2**	**1**	–	–	–	–
	–	–	–	–	**C1, C3**	**3**
	–	–	–	–	–	–
3	–	–	–	–	**C1, C2, C3**	**–0,5**

Source: own calculations.

If the total number of facilities activated is not set in advance, the optimal strategy consists in choosing all facilities except for C2. If only four facilities may be activated, A2, B1, C1 and C3 should be selected.

The problem can be expressed as

$$\sum_{j=1}^{n} c_j(p_j) \rightarrow \max \tag{6}$$

$$0 \leq p_j \leq m_j \qquad j = 1, 2, \ldots, n \tag{7}$$

$$\sum_{j=1}^{n} p_j = L \leq \sum_{j=1}^{n} m_j \tag{8}$$

where m_j is defined as the number of potential facilities in region j and L stands for the desired total number of facilities activated. The constraint (8) is optional.

The approach presented in Section 2 is relatively simple, but requires profit estimation for all combinations.

3. SECOND APPROACH – PROFIT FUNCTION FOR EACH FACILITY

In connection with the necessity of estimating the profit for plenty of variants in the first approach, the author is considering the possibility of deriving profit functions for each potential facility. Then, the maximization of their sum may constitute the objective function. The profit generated by a given facility certainly depends on its territory served called also range of coverage or influence area. Factors determining this territory can be divided into two groups. The first one consists of factors known or relatively easy to predict before the decision making (e.g. place attractiveness, population density and existing competition) and allows us to establish the original range of coverage $\left(W_{ij}^{o}\right)$. This area may be reduced after taking into account factors belonging to the second group and relating to the activation of other potential facilities from the same chain as well as to the distances between them and a particular facility. These data are known just when the optimal solution is found. The final range of coverage may be for instance defined as

$$w_{ij} = \max\left\{0, W_{ij}^{o} x_{ij} - \left(-x_{ij} + \exp\left(\sum_{l=1}^{n} \sum_{k=1}^{m_l} s_{ij}^{kl} x_{kl}\right)\right)\right\} \tag{9}$$

where x_{ij} is equal to 1 if the facility i is activated in region j and 0 otherwise, s_{ij}^{kl} is the so-called nonnegative shrinking coefficient representing the impact the activation of facility kl has on the territory served by facility ij. When facilities are far from each other the shrinking coefficient is equal to zero. The range of coverage is not treated as a circle whose radius decreases when a new facility is activated in the vicinity, because the territory served shrinks only near the new facility and not from all directions. Depending on the combination of facilities selected to be built and their location, the influence area will be equal to one of the following values:

$$\left(x_{ij} = 0\right) \Rightarrow \left(w_{ij} = 0\right) \tag{10}$$

$$\left(\left(x_{ij} = 1\right) \wedge \left(\forall x_{kl} \neq x_{ij} : x_{kl} = 0\right)\right) \Rightarrow \left(w_{ij} = W_{ij}^{o}\right) \tag{11}$$

$$\left(\left(x_{ij} = 1\right) \wedge \left(\exists x_{kl} \neq x_{ij} : x_{kl} = 1 \wedge s_{ij}^{kl} = 0\right)\right) \Rightarrow \left(w_{ij} = W_{ij}^{o}\right) \tag{12}$$

$$\left(\left(x_{ij} = 1\right) \wedge \left(\exists x_{kl} \neq x_{ij} : x_{kl} = 1 \wedge s_{ij}^{kl} > 0\right)\right) \Rightarrow \left(w_{ij} < W_{ij}^{o}\right) \tag{13}$$

The equation (9) is correct only if possible intersections of original ranges of coverage occur between two facilities (Figure 1).

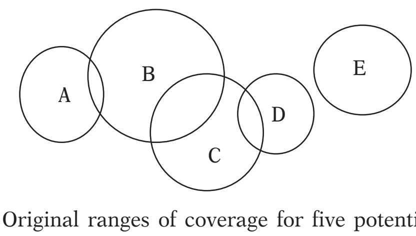

Figure 1. Original ranges of coverage for five potential facilities

Otherwise, the impact of facilities activation could be overestimated. Therefore, the formula would have to be modified by adding shrinking coefficients taking into account the simultaneous effect of the activation of more facilities.

Given the territory served by facility *ij* we can define its profit (P_{ij}) as a function of w_{ij}. The second model can emerge in the form of the following equations:

$$\sum_{j=1}^{n}\sum_{i=1}^{m_j} P_{ij}x_{ij} \to \max \tag{14}$$

$$P_{ij} = f(w_{ij}) \qquad i = 1, 2, \ldots, m_j, \quad j = 1, 2, \ldots, n \tag{15}$$

$$w_{ij} = \max\left\{0, W_{ij}^{o}x_{ij} - \left(-x_{ij} + \exp\left(\sum_{l=1}^{n}\sum_{k=1}^{m_l} s_{ij}^{kl}x_{kl}\right)\right)\right\} \qquad i = 1, 2, \ldots, m_j, \quad j = 1, 2, \ldots, n \tag{16}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} = L \le \sum_{j=1}^{n} m_j \tag{17}$$

$$x_{ij}, x_{kl} \in \{0,1\} \tag{18}$$

Since people living far from a facility will not necessarily patronize this one, it is possible to recognize that along with the growth of the range of coverage profits P_{ij} generated by a particular facility increase as well but at a decreasing speed. Thus, the author recommends applying such analytical forms for the equation (15) as hyperbola: $P_{ij} = \alpha + \beta/w_{ij}$ with a constant positive and a coefficient negative, Törnquist function: $P_{ij} = (\alpha \cdot w_{ij})/(\beta + w_{ij})$ with parameters positive, or power function: $P_{ij} = \alpha \cdot w_{ij}^{\beta}$, where α is positive and β belongs to the interval (0,1).

4. THIRD APPROACH – PROFIT FUNCTION FOR EACH MARKET AREA

The second approach does not require estimating final profits for numerous combinations, but establishing a suitable analytical form and parameters for each territory served and profit function, which may cause difficulties. Therefore, one more algorithm with the following steps is suggested:

1. Specify the area with potential facilities.
2. Split this area into K smaller market areas (Figure 2).

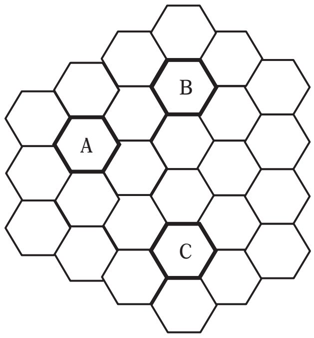

Figure 2. Splitted area with potential facilities

3. Determine for each potential facility the original range of coverage as a set of market areas (Figure 3).

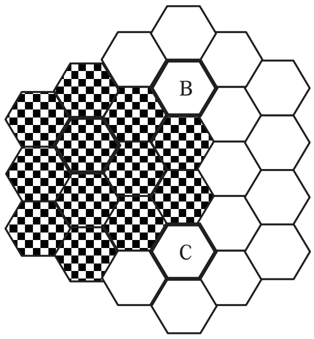

Figure 3. Facility A: the original range of coverage

4. Estimate profits generated by potential facilities from each market area assuming that only one facility will be activated.

5. Distinguish those market areas which belong to more than one original territory served (Figure 4).

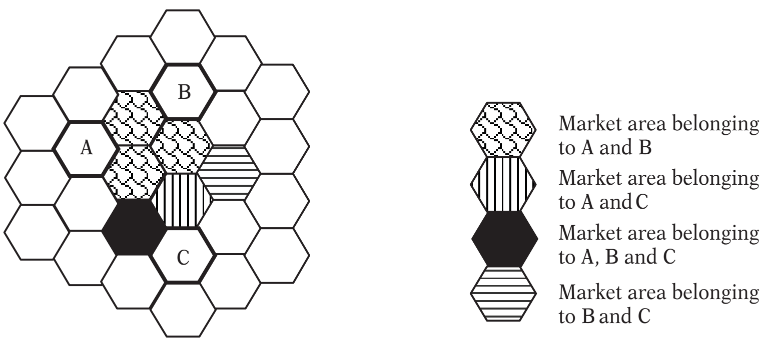

Figure 4. Market areas belonging to more than one original territory served

6. Estimate shares in profits when more than one facility takes advantages of a particular market area.

7. Using a suitable optimization model find the set of L facilities maximizing the total profit generated from all market areas.

It is recommended to present the split area as a hexagonal honeycomb proposed by A. Lösch [6], because hexagons cover the area without leaving holes. Steps 3 and 5 allow us to obtain a diagram similar to a Voronoi diagram, also called Dirichlet tessellation [15]. Using Euclidean distance as the only criterion to delimit territories served for each facility is impossible, because customers choose the store considering a trade-off between proximity and store attractiveness [7]. That is why the diagram does not resemble an ordinary Voronoi diagram, but a multiplicatively weighted Voronoi diagram. The third optimization model is presented below:

$$\sum_{k=1}^{K} h_k \to \max \tag{19}$$

$$h_k = \begin{cases} 0 & \text{if} \quad \forall x_{ij}^k : x_{ij}^k = 0 \\ \sum_{j=1}^{n} \sum_{i=1}^{m_j} h_{ij}^k s_{ij}^k x_{ij}^k & \text{if} \quad \exists x_{ij}^k : x_{ij}^k = 1 \end{cases} \quad k = 1, 2, \ldots, K \tag{20}$$

$$\begin{gathered} s_{ij}^k = f\left(x_{11}^k, x_{21}^k, \ldots, x_{m_1 1}^k, x_{12}^k, x_{22}^k, \ldots, x_{m_2 2}^k, \ldots, x_{m_n n}^k\right) \\ i = 1, 2, \ldots, m_j, \; j = 1, 2, \ldots, n, \; k = 1, 2, \ldots, K \end{gathered} \tag{21}$$

$$x_{ij} = \begin{cases} 0 \;\; \text{if} \quad \sum_{k=1}^{K} x_{ij}^k = 0 \\ 1 \;\; \text{if} \quad \sum_{k=1}^{K} x_{ij}^k > 0 \end{cases} \quad i = 1, 2, \ldots, m_j, \; j = 1, 2, \ldots, n \tag{22}$$

$$\sum_{j=1}^{n} \sum_{i=1}^{m_j} x_{ij} = L \le \sum_{j=1}^{n} m_j \tag{23}$$

$$s_{ij}^k \in [0, 1] \tag{24}$$

$$x_{ij}, x_{ij}^k \in \{0, 1\} \tag{25}$$

where h_k means the estimated profit generated by the market area k. The variable x_{ij}^k is equal to 1 when the facility ij, whose original territory served covers this market area, is activated. The estimated profit generated from the market area k by the facility ij being the only one taking advantages of this market area is given by h_{ij}^k and s_{ij}^k, signifies facility share in profits from the market area. The variable x_{ij} is equal to 1 if the facility i is activated in region j and 0 otherwise.

Notice that shares in profits depend on the set of facilities activated. Assume that the market area $k = 1$ belongs to the original range of coverage of two potential facilities: ($i = 1$, $j = 1$) and ($i = 2$, $j = 1$). If only one of them is built, the estimated profit is equal to $h_{11}^1 = 7$ or $h_{21}^1 = 6$. If the two facilities are activated, shares will amount to $s_{11}^1 = 3/4$ and $s_{21}^1 = 1/2$. For the case presented above, the equation (20) comes in the form:

$$h_1 = \begin{cases} 0 & \text{if } \; x_{11}^1 = 0, x_{21}^1 = 0 \\ 7 \cdot 1 \cdot x_{11} & \text{if } \; x_{11}^1 = 1, x_{21}^1 = 0 \\ 6 \cdot 1 \cdot x_{21} & \text{if } \; x_{11}^1 = 0, x_{21}^1 = 1 \\ 7 \cdot (3/4) \cdot x_{11} + 6 \cdot (1/2) \cdot x_{21} & \text{if } \; x_{11}^1 = 1, x_{21}^1 = 1 \end{cases} \tag{26}$$

The sum of shares for a given market area does not need to be equal to 1, because shares signify a part of profits generated by a particular facility and not a part of the profit achieved together. When the area is served by more than one facility, the global profit from this area is usually higher than the profit generated from it by the only facility activated.

5. PARAMETERS ESTIMATION

The application of models presented above is quite simple apart from parameters estimation which concerns the territory served as well as profits and shares in profits. There are parameters that should describe the influence of exogenous factors relating to the population, place attractiveness, existing competition and accessible means of transport (e.g. W_{ij}^o, h_{ij}^k) and parameters showing how a given facility may depend on other facilities from the same chain (e.g. s_{ij}^{kl}, s_{ij}^k). Notice that depending on how travel time or distances are measured parameters amount to different values. Ponsard [9] suggests applying an Euclidean, rectangular, peripherical distance or a network, but others recommend especially the last measure saying that human mobility is usually limited by the transport network. In addition, a network space, in contrast to Euclidean or rectangular distances, reduces the problem to one dimension [2]. Models presented by Sadahiro [10], Dasci and Laporte [4] may be also very useful. Sadahiro states that the accessibility and concentration measures may be established on the basis of opening hours and locations of each facility, which may help to estimate shares in profits. Dasci and Laporte show how to find the market boundary for two neighbor stores under the assumption that customers patronize that one which provides the minimum delivered price (products' price + cost of transport). Many other approaches relating to patronage probability calculation are presented in literature (e.g. [1], [7]).

6. CONCLUSION

This paper presents three possible approaches concerning the optimal location of homogeneous facilities belonging to one proprietor. In the first approach it is demonstrated that methods used in resources allocation problem may also be applied to location issues but only if interactions occur between facilities built in the same region. The second one emphasizes the range of coverage as an important factor determining profits generated by facilities, and in the third approach Lösch's theory and Voronoi diagrams are used to maximize the total profit from market areas. Depending on available data, one of these models can be applied. They allow taking into consideration interactions between facilities and they are quite simple. Moreover, models are flexible because their equations may have other analytical forms than those presented in the paper. In the further investigation that could be carried out, parameters may be replaced with random variables with known distribution. A natural extension is to enrich the research by a temporal facet where cumulative values of parameters are estimated for different periods. Such an approach allows to consider the evolution of phenomena influencing profits generated by a facility as well as taking into account new significant factors that may appear in the future (e.g. new competition). Such an approach also allows to establish an optimal location strategy for a given moment.

Uniwersytet Ekonomiczny w Poznaniu

REFERENCES

[1] Brandeau M.L., Chiu S.S., [1994], *Location of competing facilities in useroptimizing environment with market externalities*, TS, 28 (2), 125-140.
[2] Brimberg J., Korach E., Eden-Chaim M., Mehrez A., [2001], *The capacited p-facility location problem on the real line*, „International Transactions of Operational Research", 8, 727-738.
[3] Brimberg J., Love R.F., [1998], *Solving a class of two-dimensional uncapacited location-allocation problems by dynamic programming*, „Operations Research", 46, 702-709.
[4] Dasci A., Laporte G., [2004], *Location and pricing decisions of a multistore monopoly in a spatial market*, „Journal of Regional Science", 44, 489-515.
[5] Hotelling H., [1929], *Stability in competition*, „Economic Journal", 39, 41-57.
[6] Lösch A., [1940], *Die räumliche Ordnung der Wirtschaft*, Jena.
[7] Mendes A.B., Themido I.H., [2004], *Multi-outlet retail site location assessment*, „International Transactions in Operational Research", 11, 1-18.
[8] Palander T., [1935], *Beiträge zur Standorttheorie*, Almqvist & Wiksell, Uppsala.
[9] Ponsard C., [1979], *Economie urbaine et espaces métriques*, „Sistemi Urbani", 1, 123-135.
[10] Sadahiro Y., [2005], *Spatiotemporal analysis of the distribution of urban facilities in terms of accessibility*, „Papers in Regional Science", 84 (1), 61-84.
[11] Samuelson W.F., Marks S.G., [2002], *Managerial Economics*, Wiley Publishing CO, New York.
[12] Steinhaus H., [1956], *Kalejdoskop Matematyczny*, PWN, Warszawa.
[13] Stevens B.H., [1961], *An application of game theory to a problem in location strategy*, „Papers and Proceedings in Regional Science Association", 7.
[14] Thünen J.H., [1826], *Der isolierte Staat In Beziehung auf Landwirtschaft und Nationalökonomie*, Hamburg.
[15] Voronoi G.F., [1908], *Nouvelles applications des paramètres continus a la théorie des formes quadratiques. Deuxième mémoire. Recherche sur les paralléloèdres primitifs*, „Journal für Reine und Angewandte Mathematik", 134, 198-287.

Praca wpłynęła do redakcji w lipcu 2009 r.

DYSKRETNE ZAGADNIENIE LOKALIZACYJNE DLA SIECI JEDNORODNYCH OBIEKTÓW

Streszczenie

Autorka pracy wymienia możliwe kryteria podziału zagadnień lokalizacyjnych opisanych w literaturze, a następnie przedstawia propozycję trzech dyskretnych modeli optymalizacyjnych, które mogą znaleźć zastosowanie przy opracowywaniu projektu uruchomienia sieci jednorodnych obiektów należących do jednego właściciela. Celem zadań jest maksymalizacja całkowitego zysku osiągniętego przez tegoż właściciela, a nie maksymalizacja zysku zrealizowanego przez każdy obiekt osobno. Autorka ukazuje powiązanie pierwszego modelu z zagadnieniem alokacji zasobów. Wpływ odległości pomiędzy obiektami na obszar zasięgu poszczególnych obiektów został w szczególności uwzględniony w drugim i trzecim modelu optymalizacyjnym. W ostatnim zadaniu wykorzystano założenia Löscha i Voronoi.

Słowa kluczowe: Zagadnienie lokalizacyjne, Dyskretne modele optymalizacyjne, Maksymalizacja zysku, Jednorodne obiekty, Obszar zasięgu, Diagram Voronoja (tesselacja Dirichleta), Zagadnienie rozdziału zasobu, Metoda ekstremów lokalnych, Metoda zysków krańcowych.

THE DISCRETE LOCATION PROBLEM FOR A CHAIN OF HOMOGENEOUS FACILITIES

Summary

The beginning of the article is devoted to a review of different location problems discussed in the literature. In the main part of this contribution the author presents and compares three discrete optimization models that may be useful for decision-makers considering the construction and activation of a chain of homogeneous facilities belonging to one proprietor. The models goal is to maximize his or her total profit and not the gain of each facility separately. The author shows the connection of the first model with the resources allocation problem. The influence of the distance between facilities on their territory served is emphasized particularly in the second and third approach. The last model is partially based on Lösch's and Voronoi's principles.

Key words: Location problem, Discrete optimization models, Profit maximization, Homogeneous facilities, Territory served (range of coverage, influence area), Voronoi diagram (Dirichlet tessellation), Resources allocation problem, Local extrema method, Marginal profits method.