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ŁUKASZ KWIATKOWSKI*

MARKOV SWITCHING SV PROCESSES IN MODELLING VOLATILITY OF FINANCIAL TIME SERIES[1]

1. INTRODUCTION

It is well-known that the volatility of financial time series tends to change over time. In modelling time-variable nature of volatility the family of ARCH processes, introduced by Engle (see [8]), has been an evident breakthrough. The success of the above was followed by their numerous generalizations, with already 'classical' GARCH processes of Bollerslev (see [3]) among many. The common feature of these autoregressive conditional heteroscedastic constructions is that the conditional variance – as a measure of asset volatility and uncertainty – depends entirely on the past information of the data generating process. This implies that the evolution of conditional volatility is determined only by the past changes of the asset price. An alternative model, proposed by Clark in [6], assumes that the volatility is a separate latent process, which is attained by introduction of a separate innovation term into the log-volatility equation following a simple AR(1) process. This model is widely known in the literature as a Stochastic Volatility (SV) model. Both GARCH and SV processes have been receiving much attention in the literature and gained unequalled popularity among researchers.

The underlying assumption of the above parametric models is that there are no structural shifts throughout the period over which data is analyzed. This allows the researcher to presume that all of the estimated parameters of the model are constant over time. However, volatility clustering, a common phenomenon observed in stock returns series, may question this belief. It is so due to some heuristic reasoning that less volatile periods alternating with these of higher uncertainty may somehow correspond with structural breaks occurring in the data. In view of potential heterogeneity of a certain time series, models such as GARCH or SV are of too restrictive nature (see [12] and [13]). Not being able to capture discrete shifts of the states of the economy may be the cause for these models to yield somewhat misleading results. For instance, Granger and Hyung in [10] and Diebold and Inoue in [7] suggest that structural

* Assistant at The Andrzej Frycz Modrzewski Kraków University College, Department of Statistics.
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breaks in the mean of volatility may be a source of volatility persistence. It follows that a proper model should include an explicit mechanism capable of accounting for possible regime changes. One of the most popular in this regard is a Markov switching (MS) mechanism introduced by Hamilton in [11]. What he suggested is an autoregressive process whose parameters are subject to changes over time according to a latent discrete homogeneous Markov chain. Since then many studies have been undertaken to employ the idea of MS into volatility models, mainly those of the GARCH family (see [2], among many).

This paper aims to present generalizations of the Markov Switching Stochastic Volatility process introduced by So *et al.* in [19]. Their model allows for only one of the three parameters in the log-volatility equation to switch between a predetermined number of states. In our study we generalize the model studied by So *et al.* in [19] by allowing all the volatility parameters to switch over the regimes. In the model proposed by Hwang *et al.* in [12] and [13] all three parameters, i.e. the intercept, the elasticity of volatility and the volatility of volatility are state-dependent[2]. However, their specification of the log-volatility equation slightly differs from the one employed by So *et al.* in [19] and, hence, their model cannot be viewed as a generalization of the latter.

There have been a few estimation methods of the MSSV constructions, both in the 'classical' and Bayesian setting. For instance, So *et al.* in [19] develop Gibbs sampling algorithm for a *K*-state MSSV model with a switching volatility intercept. Other Bayesian estimation procedures, based on particle filter technique, are presented by Shibata and Watanabe in [17], Casarin in [5] and Carvalho and Lopes in [4]. These methods are usually computationally intensive and so a simpler approach, based on the quasi-maximum likelihood technique, was proposed by Smith in [18] and used also by Hwang *et al.* in [12] and [13]. In our study we follow the latter.

The structure of the article is as follows. In Section 2 we define a two-state MSSV process and give some account of the sample properties of simulated paths. We also provide analytical formulas for its conditional (upon the current regime) and unconditional expectation and variance of the underlying log-volatility process. In Section 3 we describe the QML estimation procedure. The methodology is applied to the data from the Polish stock market. Specifically, we analyze the 1-month Warsaw Interbank Offered Rate (WIBOR1M) interest rates over the period from January 3, 2000 to December 18, 2007. The results are reported in Section 4. Finally, Section 5 concludes.

2. TWO-STATE MARKOV SWITCHING SV MODEL (MSSV)

Let S_t denote the number of the 'present' (i.e. at time t) state of the system. Here we allow the system to switch over only two states, and so S_t may equal either

[2] Term of 'elasticity of volatility' is used by Smith in [18] with reference to the autoregression parameter, φ, in the log-volatility equation of a SV model: $\ln h_t = \mu + \varphi \ln h_{t-1} + \sigma\eta_t$. Assuming $\eta_t \sim iiN(0,1)$ (i.e. each random variable η_t is identically, independently and normally distributed with zero mean and unit variance) the third parameter, σ, is a standard deviation of the innovation term $\sigma\eta_t$, and hence referred to as 'volatility of volatility'.

0 or 1 for each $t \in N \cup \{0\}$[3]. We define transition probabilities p_{ij} from state j to i as $p_{ij} = \Pr(S_t = i | S_{t-1} = j)$ where i, j are either 0 or 1. Since we assume the process governing regime changes to be a first-order homogenous Markov chain, transition probabilities are constant over time. The corresponding transition matrix, P, is defined as

$$P = \begin{bmatrix} p_{00} & p_{01} = 1 - p_{11} \\ p_{10} = 1 - p_{00} & p_{11} \end{bmatrix}.$$

The stochastic process for S_t is strictly stationary and admits the following AR(1) representation (see [11]):

$$S_t = 1 - p_{00} + (-1 + p_{00} + p_{11})S_{t-1} + V_t, \tag{1}$$

where a non-typical distribution of the disturbance term V_t is specified as in [11].

In the further analysis the unconditional (upon the previous state) probabilities of the system being in one of the two states will be of use, that is $\Pr(S_t = 0)$ and $\Pr(S_t = 1)$. From the basic Markov chain theory it is known that to define a homogenous two-state Markov process it is enough to define these probabilities at time $t = 0$ and the transition probabilities p_{ij}. Denoting vector of unconditional probabilities $[\Pr(S_t = 0)\ \Pr(S_t = 1)]$ at time t as $x^{(t)}$, and given the transition matrix P, the following recursive relation is valid:

$$x^{(t)} = x^{(t-1)} P' = x^{(0)} (P')^t,$$

where the apostrophe is the matrix transpose operator. From the above it is seen that unconditional probabilities in $x^{(t)}$ are time-varying. However, assuming that the process has begun in the indefinite past and the chain is ergodic (which here is the case when $\forall_{i,j \in \{0,1\}}\ p_{ij} \in (0,1)$), the $x^{(t)}$ converges with $t \to \infty$ to the limiting vector π of the ergodic probabilities, that is:

$$\lim_{t \to \infty} x^{(t)} = \begin{bmatrix} p_0 & p_1 \end{bmatrix} = \pi,$$

which are constant over time.

It is easy to show that the expectation of the process $\{S_t, t \in N \cup \{0\}\}$ following Equation (1) equals the ergodic probability of the system being in state 1, i.e.:

$$E(S_t) = \Pr(S_t = 1) = \frac{1 - p_{00}}{2 - p_{00} - p_{11}} = p_1. \tag{2}$$

Obviously we have

$$\Pr(S_t = 0) = 1 - \Pr(S_t = 1) = \frac{1 - p_{11}}{2 - p_{00} - p_{11}} = p_0.$$

[3] By 'N' we denote the set of positive integers.

The MSSV model combines the Markov switching mechanism presented above with a standard discrete-time SV process. Here we propose the following definition of a two-state MSSV process:

Definition 1

Stochastic process $\{z_t, t \in N \cup \{0\}\}$ follows a two-state Markov Switching Stochastic Volatility (MSSV) process if and only if for each $t \in N \cup \{0\}$ the following assumptions hold:

$$z_t = \varepsilon_t \sqrt{h_t}, \tag{3}$$

$$\ln h_t = \mu_0 + (\mu_1 - \mu_0)S_t + \left[\varphi_0 + (\varphi_1 - \varphi_0)S_t\right]\ln h_{t-1} + \left[\sigma_0 + (\sigma_1 - \sigma_0)S_t\right]\eta_t, \tag{4}$$

$$\left\{\begin{pmatrix}\varepsilon_t \\ \eta_t\end{pmatrix}, t \in Z\right\} \sim iiN^{(2)}\left(0_{(2\times 1)}, I_2\right), \tag{5}$$

where S_t is a random variable representing a homogenous, ergodic and irreducible two-state Markov chain with the transition probabilities, p_{ij}, defined as above.

Equation (4) defines the way of how the log-volatility process evolves over time, depending on the current regime represented by the random variable S_t. The latter is governed by an unobserved regime-switching mechanism. It is easily seen from equation (4) that we have:

$$\ln h_t = \begin{cases} \mu_0 + \varphi_0 \ln h_{t-1} + \sigma_0 \eta_t & \text{if } S_t = 0 \\ \mu_1 + \varphi_1 \ln h_{t-1} + \sigma_1 \eta_t & \text{if } S_t = 1 \end{cases}.$$

A similar model is considered by Smith in [18] and So *et al.* in [19], yet only with the intercept having the switching property. Both of the innovation terms in (3) and (4), i.e. ε_t and η_t, are identically, independently and normally distributed with zero mean and unit variance, although it may be possible to assume a different type of distribution for ε_t. The current level of volatility is determined not only by its past values and disturbances η_t, but also the current regime S_t. The random variable $\ln h_t$ is measurable with respect to the σ-field generated by the lagged $\ln h_t$'s, the current error term η_t and the current state variable S_t, so it may be shown that h_t constitutes the conditional variance of the process given by Definition 1, i.e.:

$$Var\left(z_t \middle| F_{t-1}, \eta_t, S_t\right) = h_t,$$

where F_{t-1} is the past information about the process $\{\ln h_t, t \in N \cup \{0\}\}$ up to the moment $t - 1$.

Considering different regimes of the system it is natural to characterize them in some respect. In other words, it is important to know on what account the specified regimes differ. In our setting, the two regimes are distinguished by the state-specific means and variances of the log-volatility process.

Assuming covariance stationarity[4] of the log-volatility process and based on the results of Nielsen and Olesen (see [14])[5], analytical formulas can be derived for both conditional (upon the current regime, S_t) and unconditional means and variances of the log-volatility process. The conditional means of $\ln h_t$ upon the states $S_t = 0$ and $S_t = 1$ are given by:

$$E\left(\ln h_t \middle| S_t = 0\right) = \frac{\mu_0(1-\varphi_1 p_{11}) + \mu_1\varphi_0(1-p_{00})}{1-\varphi_0\varphi_1 - \varphi_1 p_{11}(1-\varphi_0) - \varphi_0 p_{00}(1-\varphi_1)} \tag{6}$$

and

$$E\left(\ln h_t \middle| S_t = 1\right) = \frac{\mu_1(1-\varphi_0 p_{00}) + \mu_0\varphi_1(1-p_{11})}{1-\varphi_0\varphi_1 - \varphi_1 p_{11}(1-\varphi_0) - \varphi_0 p_{00}(1-\varphi_1)}, \tag{7}$$

respectively.

The following relation between the conditional and unconditional expectations of $\ln h_t$ is straightforward:

$$E(\ln h_t) = \sum_{j=0}^{1} p_j E\left(\ln h_t \middle| S_t = j\right),$$

with p_j ($j = 0, 1$) being the ergodic probabilities of the unobserved Markov chain.

It is worth noticing that both state-dependent means are functions of the switching intercept as well as the regime-changing elasticity of volatility. In our belief this may imply that whenever periods of different volatility level occur in the analyzed time series (which corresponds with volatility clustering), it may results from regime-shifts in the intercept or in the persistence parameter or, eventually, in both of them simultaneously.

It can be shown that formulas for the second-order conditional moments of $\ln h_t$ (generally of a switching AR(1) process, see [14]) are given by:

$$E\left((\ln h_t)^2 \middle| S_t = 0\right) = \frac{d_0(1-\varphi_1^2 p_{11}) + d_1\varphi_0^2(1-p_{00})}{1-\varphi_0^2 p_{00} - \varphi_1^2 p_{11} + \varphi_0^2\varphi_1^2(-1+p_{00}+p_{11})} \tag{8}$$

and

$$E\left((\ln h_t)^2 \middle| S_t = 1\right) = \frac{d_1(1-\varphi_0^2 p_{00}) + d_0\varphi_1^2(1-p_{11})}{1-\varphi_0^2 p_{00} - \varphi_1^2 p_{11} + \varphi_0^2\varphi_1^2(-1+p_{00}+p_{11})}, \tag{9}$$

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[4] Both covariance and strict stationarity conditions for a general class of multivariate Markov-switching ARMA models are studied by Francq and Zakoian in [9]. Based on their results, the necessary and sufficient condition for covariance stationarity of the log-volatility process presented by Equation (4) is:

$$\begin{cases} p_{00}\varphi_0^2 + p_{11}\varphi_1^2 + (1-p_{00}-p_{11})\varphi_0^2\varphi_1^2 < 1, \\ p_{00}\varphi_0^2 + p_{11}\varphi_1^2 < 2. \end{cases}$$

It may be shown that if $\varphi_0 = \varphi_1$, then the above condition simply reduces to $|\varphi| < 1$.

[5] The authors consider a Markov-switching AR(1) process and so their results are easily adopted to the log-volatility equation of the same switching AR(1) form.

where

$$d_i = \mu_i^2 + 2\mu_i\varphi_i E\left(\ln h_{t-1} \middle| S_t = i\right) + \sigma_i^2, \quad i = 0,1. \tag{10}$$

The expectation in Equation (10) is computed as

$$E\left(\ln h_{t-1} \middle| S_t = i\right) = \sum_{j=0}^{1} p_{ji}^{*} E\left(\ln h_t \middle| S_t = j\right)$$

with $p_{ji}^{*} = \Pr\left(S_{t-1} = i \middle| S_t = j\right)$ being the *inverse* transition probabilities, which – in the case of a two-state Markov chain – can be easily shown to equal the ordinary transition probabilities, p_{ij}. Then the unconditional second-order moment is obtained according to the formula:

$$E\left((\ln h_t)^2\right) = E_{S_t}\left[E\left((\ln h_t)^2 \middle| S_t\right)\right] = \sum_{i=0}^{1} P(S_t = i) \cdot E\left((\ln h_t)^2 \middle| S_t = i\right) = \sum_{i=0}^{1} p_i \cdot E\left((\ln h_t)^2 \middle| S_t = i\right).$$

Computation of the state-specific and unconditional variances of the log-volatility process is straightforward:

$$Var\left(\ln h_t \middle| S_t = i\right) = E\left((\ln h_t)^2 \middle| S_t = i\right) - E^2\left(\ln h_t \middle| S_t = i\right)$$

and

$$Var(\ln h_t) = E\left((\ln h_t)^2\right) - E^2(\ln h_t),$$

respectively.

The MSSV process given by Definition 1 encompasses the following special cases:

– when $\varphi_0 = \varphi_1$ and $\sigma_0 = \sigma_1$ we have a MSSV model with only the volatility intercept being regime-dependent; we shall refer to this specification as MSSV(μ),

– when $\mu_0 = \mu_1$ and $\sigma_0 = \sigma_1$ we have a MSSV model with switching the autoregression parameter only; we shall refer to this specification as MSSV(φ),

– when $\mu_0 = \mu_1$ and $\varphi_0 = \varphi_1$ we have a MSSV model with switching the volatility of volatility only; we shall refer to this specification as MSSV(σ),

– when $\mu_0 = \mu_1$ and $\varphi_0 = \varphi_1$ and $\sigma_0 = \sigma_1$ the process reduces to a basic stochastic volatility model (BSV); however, transition probabilities p_{ij} are then unidentified.

Each of the above is of interest in Section 4, where empirical results for the Polish stock market data are reported.

We close this section with presentation of a simulated path of a particular MSSV process along with the sample distributions of the generated data and the log-volatilities, $\ln h_t$'s.

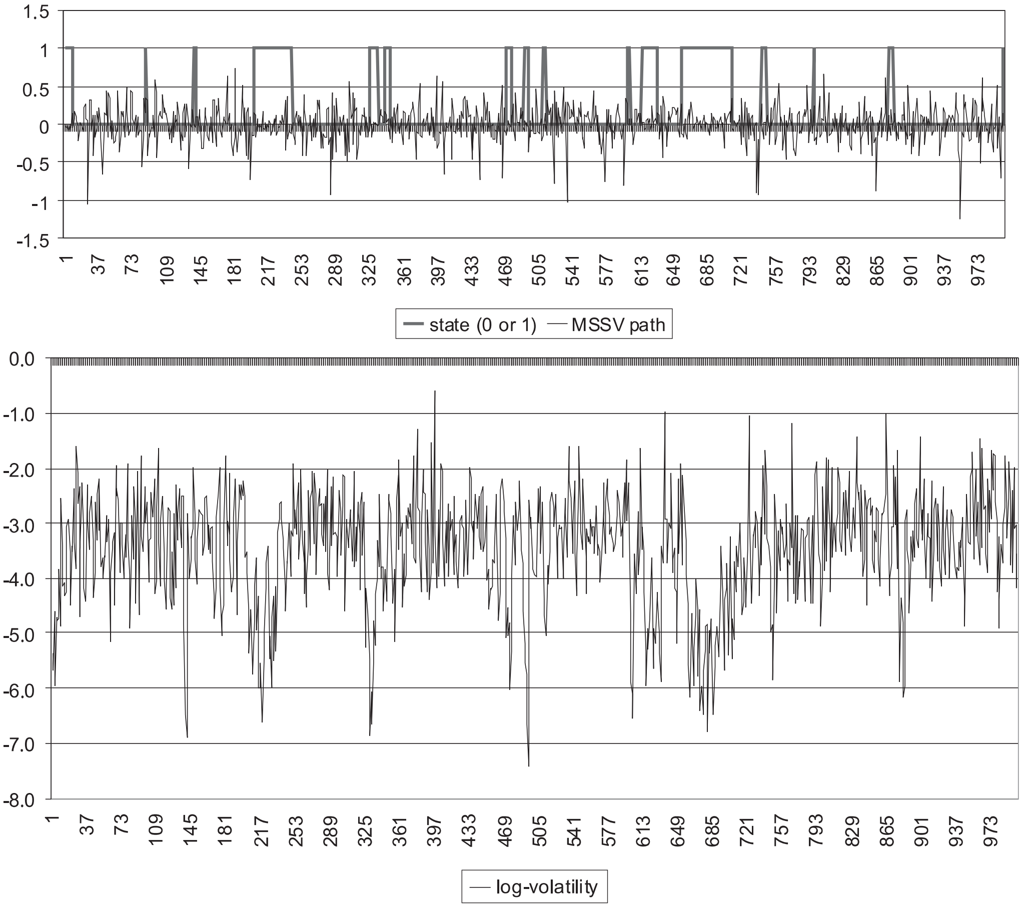

Figure 1. Simulated path of the MSSV(φ) process ($\mu = -2.5$; $\varphi_0 = 0.2$; $\varphi_1 = 0.5$; $\sigma^2 = 0.6132$; $p_{00} = 0.98$; $p_{11} = 0.95$) and the corresponding regime-switching process (top) and the log-volatility process (bottom)

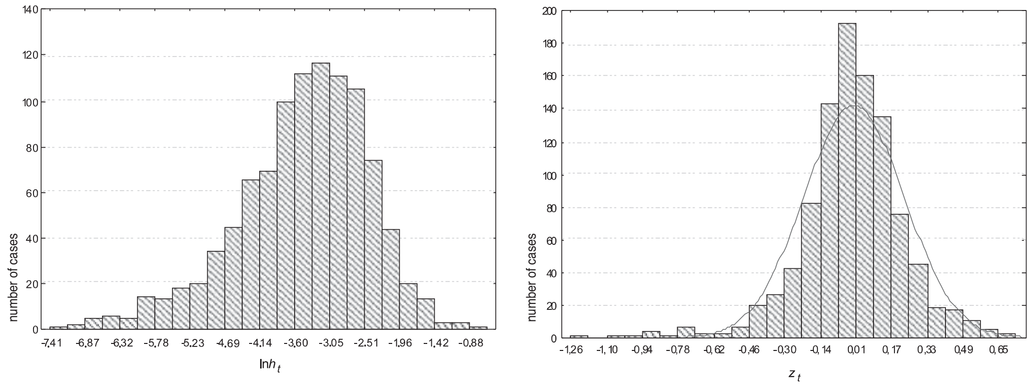

Figure 2. Sample distributions of the log-volatilities, $\ln h_t$'s, (left) and the data (right; normal distribution fitted) simulated from the MSSV process depicted in Figure 1, z_t

It is easily seen that allowing the autoregression parameter to switch over the regimes results in occurring shifts in the mean of the log-volatility process. This appears in the simulated data as the common financial data phenomenon known as *volatility clustering* (see Figure 1).

Sample distribution of the generated log-volatility process is negatively skewed. The asymmetry results from the switches of the process – from the predominant high-volatility regime ($S_t = 0$) to the low-volatility regime ($S_t = 1$). Simulated data displays evident non-normality due to relatively higher concentration around zero and 'fat tails', which is typical of most financial time series.

3. ESTIMATION PROCEDURE

Estimation of the MSSV models is not trivial. In this study we use the QML estimation technique[6] combining the standard Kalman filter and Hamilton's filter. This procedure, presented by Smith in [18] and used by Hwang *et al.* in [12] and [13], requires transforming the MSSV model given by Definition 1 into a state-space form, which is achieved by squaring and making logarithm of Equation (3). Upon the following notation:

- $\ln\left(z_t^2\right) - E\left(\ln \varepsilon_t^2\right) = y_t^*$,
- $\ln \varepsilon_t^2 - E\left(\ln \varepsilon_t^2\right) - \xi_t$,
- $\ln h_t \equiv x_t$,

we obtain the required state-space representation of the MSSV process:

$$y_t^* = x_t + \xi_t, \tag{11}$$

$$x_t = \mu_0 + (\mu_1 - \mu_0)S_t + \left[\varphi_0 + (\varphi_1 - \varphi_0)S_t\right]x_{t-1} + \left[\sigma_0 + (\sigma_1 - \sigma_0)S_t\right]\eta_t, \tag{12}$$

where ξ_t is an error term of the 'observation equation' (11), distributed as a log-chi-squared random variable with one degree of freedom, zero mean and the variance equal to $\pi^2/2$ (see [16])[7]. However, we shall treat ξ_t as if it were normally distributed with the same mean and variance, although in [12] the latter is subject to estimation along with the other parameters of model (11)-(12). Equation (12) is also referred to as the 'state equation', as it defines the unobserved log-volatility process.

The basic concept of the QML technique is that both x_t and conditional probabilities $\Pr\left(S_t = i, S_{t-1} = j \,\middle|\, \Psi_{t-1}\right)$, where $\Psi_{t-1} = \left\{y_1^*, y_2^*, \ldots, y_{t-1}^*\right\}$ constitutes the information set available at time $t-1$, are unobserved processes which can be obtained through predicting and updating procedure proposed by Smith in [18]. The Kalman filter technique is applied to forecast x_t upon the information set Ψ_{t-1} and then the forecast is updated with new information once y_t^* is available. A slightly modified Hamilton's

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[6] The QML procedure was also used by Ruiz (see [16]) and Pajor (see [15]) to estimate simple SV models.

[7] It can be shown that if $\varepsilon_t \sim iiN(0,1)$ then the expectation and the variance of a random variable ε_t^2 are -1.27 (approximately) and $\pi^2/2$, respectively (see [1]).

filter (see [11]) is employed to 'retrieve' latent conditional probabilities from the data. For a detailed presentation of the algorithm we refer to [18].

One of the product of the filter is the conditional log-likelihood function calculated as

$$\ln L(\theta; y^*) = \frac{1}{T}\sum_{t=1}^{T}\ln f\left(y_t^* \middle| \Psi_{t-1}; \theta\right), \tag{13}$$

where T is the number of the observations, $\theta = (\mu_0\ \mu_1\ \varphi_0\ \varphi_1\ \sigma_0\ \sigma_1\ p_{00}\ p_{11})'$ is a vector of the estimated parameters[8] and function $f(.)$ is the likelihood function defined as in [18]. To obtain the QML estimate of the parameter vector, θ, we maximize (13) numerically[9] with respect to θ, i.e.:

$$\hat{\theta} = \arg\max_{\theta \in \Theta}\left(\ln L(\theta; y^*)\right).$$

As noted by Smith in [18] and Hwang *et al.* in [12], the log-likelihood function of MSSV models has many local maxima and thus it is required to start optimization procedure from different sets of starting values. Once the estimate of θ has been obtained, estimate of the asymptotic covariance matrix and standard errors are calculated according to the formulas:

$$\hat{V}_{as}(\hat{\theta}) = \frac{1}{T}A_T(\hat{\theta})^{-1}B_T(\hat{\theta})A_T(\hat{\theta})^{-1},$$

where

$$A_T(\hat{\theta}) = \left\{-\frac{1}{T}\sum_{t=1}^{T}\frac{\partial^2 \ln f\left(y_t^* \middle| \Psi_{t-1}; \theta\right)}{\partial \theta_i\, \partial \theta_j}\right\}\Bigg|_{\theta = \hat{\theta}}$$

and

$$B_T(\hat{\theta}) = \left\{\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ln f\left(y_t^* \middle| \Psi_{t-1}; \theta\right)}{\partial \theta_i} \cdot \frac{\partial \ln f\left(y_t^* \middle| \Psi_{t-1}; \theta\right)}{\partial \theta_j}\right\}\Bigg|_{\theta = \hat{\theta}}.$$

The gradients and Hessian matrices in the above formulas are obtained numerically using the built-in procedures in Gauss 8.0.

4. EMPIRICAL STUDY

4.1. DATA

We analyze a total number of 2002 observations on daily log-returns on the 1-month Warsaw Interbank Offered Rate (WIBOR1M) interest rates, defined as:

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[8] Since $p_{10} = 1 - p_{00}$ and $p_{01} = 1 - p_{11}$, only p_{00} and p_{11} are estimated.

[9] We carry out the optimization tasks using Solver Add-in in MS Excel 2003.

$$r_t = 100 \ln (w_t/w_{t-1}),$$

where w_t denotes the asset price at time t. The sample covers the period from January 3, 2000 to December 18, 2007. The series of w_t and r_t are plotted in Figure 3 and 4, respectively.

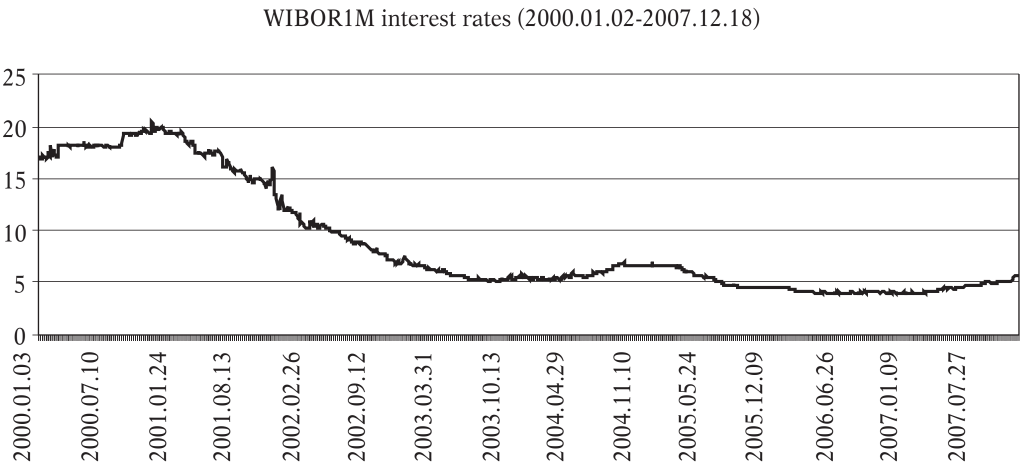

Figure 3. The series of WIBOR1M interest rates, w_t (January 3, 2000 – December 18, 2007)

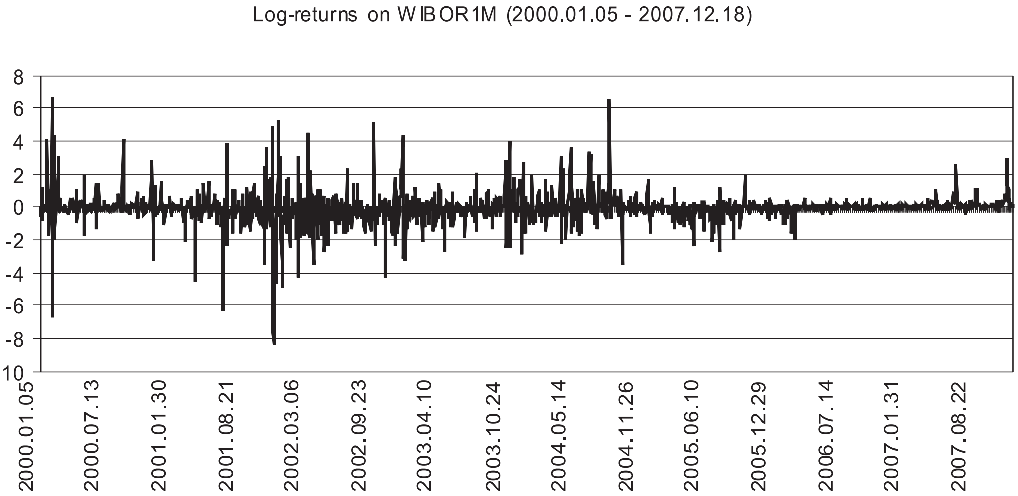

Figure 4. Daily log-returns, r_t, on the WIBOR1M (January 4, 2000 – December 18, 2007)

Further, we fit a simple AR(1) (using conditional <upon the first observation on r_t > OLS) to the series of the daily log-returns, r_t, as to filter out possible autocorrelation in the data[10]. All the following analysis is carried out for the resulting residuals[11], z_t, depicted in Figure 5 (the number of the residuals totals $T = 2000$).

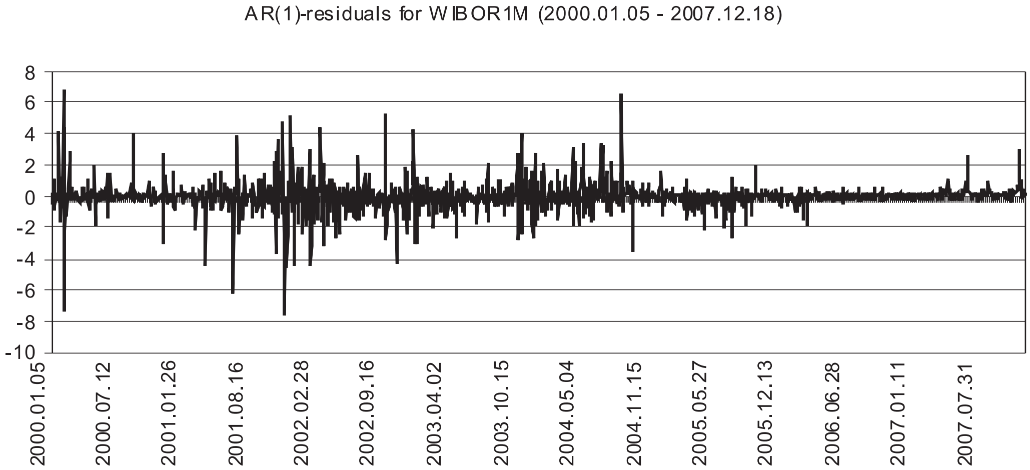

Figure 5. AR(1)-residuals for the daily log-returns on the WIBOR1M (January 5, 2000 – December 18, 2007)

As one might have expected, the series of the autoregressive residuals does not differ much from the one of the daily log-returns. Figure 5 shows the well-documented volatility clustering phenomenon in financial time series – large changes, of either sign, are typically followed by large changes, and small changes follow previous small changes. Also, one can clearly observe some changes in the volatility pattern, specifically in the latter part of the sample. This fact may be indicative of there occurring structural shifts within the analyzed period.

Table 1 contains some descriptive statistics and results of the test for the ARCH(2) effect in the series $\{z_t,\ t = 1, 2, \ldots, T\}$. The sample distribution and autocorrelation function (ACF) of the residuals and their squares are plotted in Figure 6 and 7, respectively.

Table 1

Descriptive statistics for AR(1)-residuals for WIBOR1M, z_t

Mean	Stand. deviation	Asymmetry	Kurtosis	ARCH(2)
-0.0051	0.8865	-0.3217	20.4299	$TR^2 = 206.6262$ (p-$value$ = 0.00)

[10] Fitting an AR(1) process to the data bears the results: $r_t = \underset{(0.0192)}{-0.0441} + \underset{(0.0220)}{0.1050}r_{t-1} + \hat{u}_t$.

[11] One should note that it is impossible to represent an AR(1) process with the noise term modelled as a MSSV process in a state-space form, which is required for the QML estimation procedure. Hence, we resort to that two-stage approach of fitting a simple autoregression at first and then carrying on the analysis with the residuals.

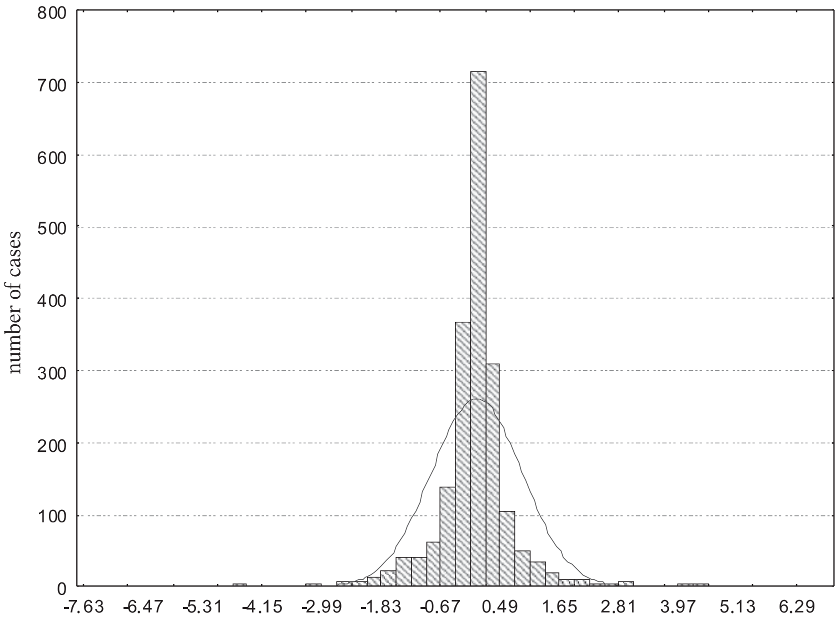

Figure 6. Sample distribution of the AR(1)-residuals for WIBOR1M

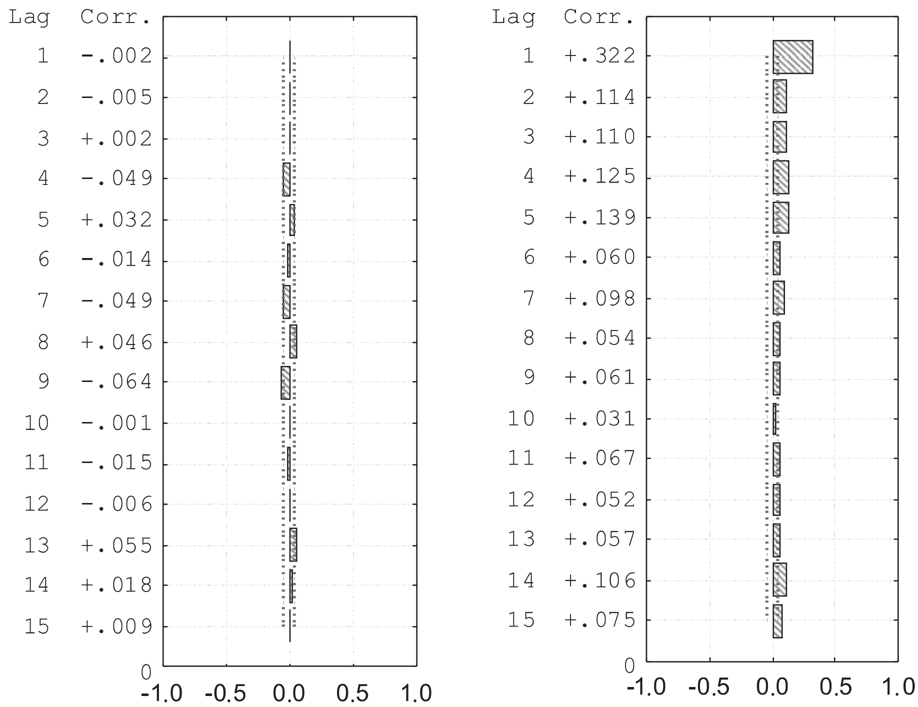

Figure 7. Sample ACF for the AR(1)-residuals (left) and their squares (right) for the WIBOR1M interest rates

The analyzed data displays typical features of financial time series, including leptokurtic empirical distribution (here also negatively skewed partly due to outliers occurring in the left tail of the distribution) and strong evidence of the ARCH effect. The autocorrelation structure of the AR-residuals and their squares also resembles the ones commonly encountered in financial data sets, i.e. hardly significant autocorrelations at any lag in the residuals accompanied with rather significant positive autocorrelations, slowly decreasing with the increase of the lag, in their squares.

4.2. EMPIRICAL RESULTS

In our paper we fit four regime-switching SV models, i.e. three specifications in which only one of the volatility parameters is allowed to regime-change (denoted as MSSV(μ), MSSV(φ), MSSV(σ)) and the general one, in which all three parameters may differ across the states (denoted as MSSV(μ, φ, σ)). We also estimate a basic stochastic volatility (BSV) model as to compare the results with the switching specifications.

The QML estimates of the parameters (along with the asymptotic standard errors) obtained for each of the considered model are reported in Table 2.

Table 2

Estimation results for AR(1)-residuals for the WIBOR1M daily log-returns

<table>
<tr><th>Parameter/model</th><th>BSV</th><th>MSSV(μ)</th><th>MSSV(φ)</th><th>MSSV(σ)</th><th>MSSV(μ, φ, σ)</th></tr>
<tr><td>μ_0</td><td rowspan="2">-0.1757*[12]
(0.0332)</td><td>-0.9047*[13]
(0.0224)</td><td rowspan="2">-0.3869*
(0.0666)</td><td rowspan="2">-0.1018
(0.0621)</td><td>-0.0288
(0.0917)</td></tr>
<tr><td>μ_1</td><td>-0.2388*
(0.0009)</td><td>-1.4248*
(0.4514)</td></tr>
<tr><td>φ_0</td><td rowspan="2">0.9113*
(0.0166)</td><td rowspan="2">0.6924*
(0.0018)</td><td>0.8515*
(0.0158)</td><td rowspan="2">0.9480*
(0.0296)</td><td>0.9854*
(0.0440)</td></tr>
<tr><td>φ_1</td><td>0.6147*
(0.0645)</td><td>0.3161*
(0.0924)</td></tr>
<tr><td>σ_0^2</td><td rowspan="2">0.4600*
(0.0905)</td><td rowspan="2">1.0331*
(0.0903)</td><td rowspan="2">0.8123*
(0.1898)</td><td>0.2071
(0.1578)</td><td>0.0450
(0.1598)</td></tr>
<tr><td>σ_1^2</td><td>9.4046*
(0.1257)</td><td>7.2436*
(2.7267)</td></tr>
<tr><td>p_{00}</td><td>–</td><td>0.9988*
(0.0008)</td><td>0.9990*
(0.0002)</td><td>0.9969*
(0.000023)</td><td>0.9863*
(0.0026)</td></tr>
<tr><td>p_{11}</td><td>–</td><td>0.9976*
(0.0002)</td><td>0.9982*
(0.0001)</td><td>0.9494*
(0.0012)</td><td>0.9334*
(0.0248)</td></tr>
</table>

[12] Estimates with '*' are statistically significant (significance level = 0.05). In the case of volatility parameters it means 'significantly greater than zero'.

[13] Figures in **bold** refer to switching parameters.

The results for the BSV model are quite typical, with the autoregression parameter close to one, implying strong autocorrelation in the volatility. However, if the intercept is allowed to be regime-dependent, the estimate of parameter φ is noticeably lower. It is consistent with the relevant literature (see [19] and [4], for instance). In some literature it is argued that structural breaks may generate high spurious volatility persistence, which is implied by the estimates of the misspecified Basic SV model, whereas the true persistence is far lower. However, if the volatility of volatility parameter is allowed to switch over the regimes, the persistence parameter is again close to one. It is worth noticing that in the case of each switching specification, both regime-specific parameters are markedly different from each other, which implies that there are structural breaks in the analyzed time series. Also in each of the switching models one observes estimates of probabilities of staying in a particular regime close to one. This indicates evident persistence in the Markov chain governing the regime changes, that is once the economy shifts to a particular regime little is the probability of a switch back to the previous one.

Table 3 contains the values of the maximized log-likelihood for each of the estimated models, along with the values of the Bayesian (Schwarz) and Akaike information criteria. It is seen that any switching specification is preferred to the BSV model, and the one which allows all three parameters to be state-dependent is preferred the most.

Table 3

Maximized log-likelihood and the information criteria for the estimated models

	BSV	MSSV(μ)	MSSV(φ)	MSSV(σ)	MSSV(μ, φ, σ)
$\ln L(\hat{\theta}; y^*)$	-4776.5206	-4747.6032	-4761.2629	-4759.9126	-4732.4625
AIC	9559.0411	9507.2064	9534.5257	9531.8253	**9480.9249**
BIC	9575.8438	9540.8119	9568.1312	9565.4307	**9525.7322**

Finally, to compare *in-sample* performance of the considered models, in Table 4 we report some descriptive statistics and the results of the test for the ARCH(2) effect obtained for the standardized residuals. Residual means and standard deviations are fairly the same for each of the models and equal approximately 0.05-0.06 and 1.10-1.26, respectively. It is seen that only the residuals of the most general MSSV model display no asymmetry, whereas the least residual kurtosis is generated by the model allowing only the intercept to change over the regimes. Although in each case the kurtosis coefficient is markedly lower than the one for the modelled data (see Table 1), still there is evidence of leptokurtosis of the sample distribution unexplained by the model. The latter may suggest assuming a *t*-Student distribution for the error term in Equation (3) (see Definition 1). However, it is beyond the scope of the present study. As one could have expected, no significant ARCH effect is present in any of the residual series.

Table 4

Descriptive characteristics of the standardized residuals from the estimated models

	BSV	MSSV(μ)	MSSV(φ)	MSSV(σ)	MSSV(μ, φ, σ)
Mean	0.0650	0.0590	0.0637	0.0505	0.0493
Stand. deviat.	1.2549	1.1038	1.1729	1.2559	1.1704
Asymmetry	-0.2848	-0.2733	-0.1620	-0.2398	**-0.0395**
Kurtosis	12.3094	**8.0004**	8.6170	13.6949	10.0033
ARCH(2) (*p-value*)	0.0698 (0.9657)	0.3005 (0.8605)	0.2161 (0.8976)	0.2861 (0.8667)	0.9139 (0.6332)

Below we present the smoothed probabilities $\Pr(S_t = 1|\Psi_T)$ and the corresponding smoothed log-volatilities[14] for each of the models (see Figures 8-11). Table 5 contains the estimates of the regime-specific and unconditional characteristics of the log-volatility process.

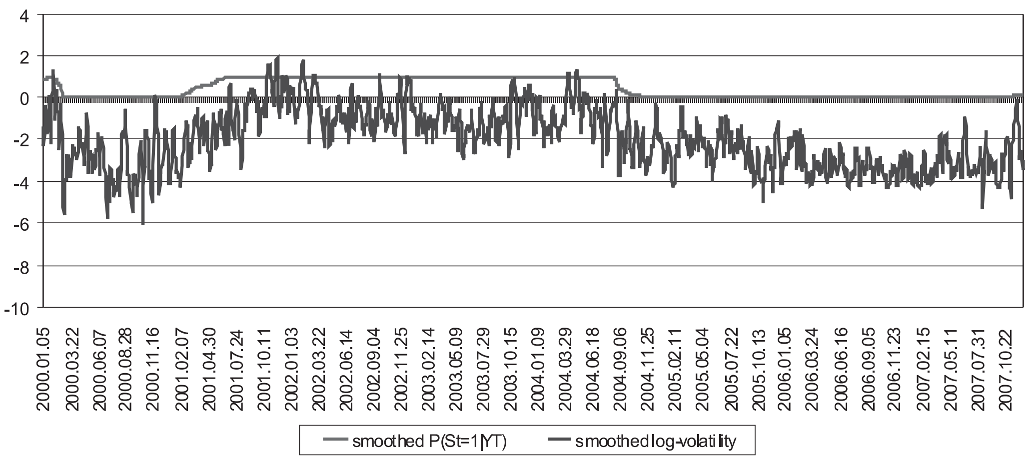

Figure 8. Smoothed probabilities $\Pr(S_t = 1|\Psi_T)$ and log-volatilities for MSSV(μ)

[14] See [18] for a detailed presentation of the smoothing algorithm.

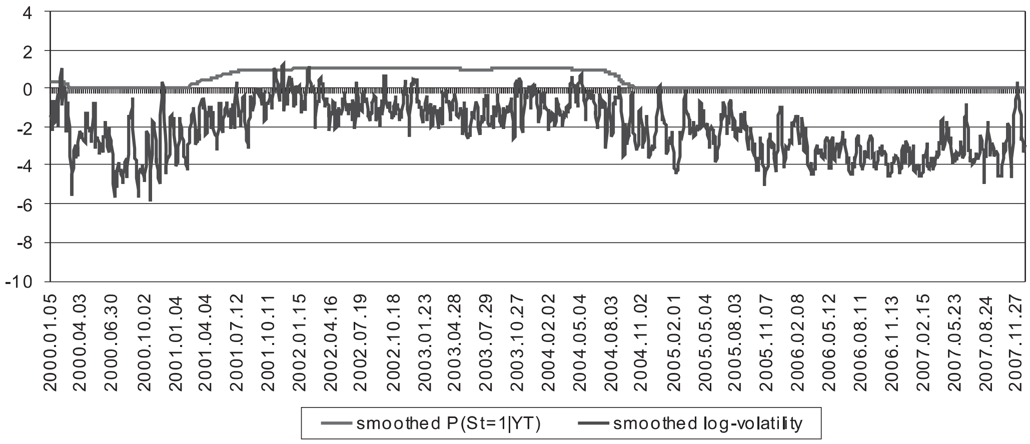

Figure 9. Smoothed probabilities $\Pr(S_t = 1|\Psi_T)$ and log-volatilities for MSSV(φ)

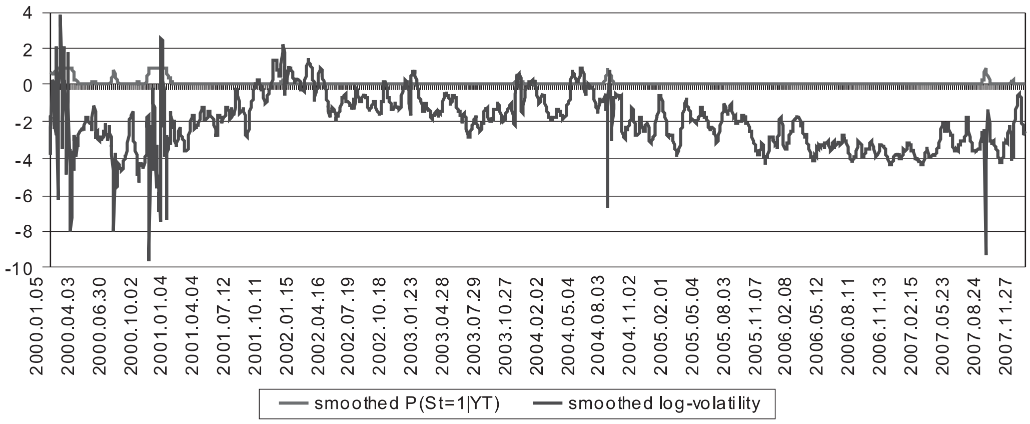

Figure 10. Smoothed probabilities $\Pr(S_t = 1|\Psi_T)$ and log-volatilities for MSSV(σ)

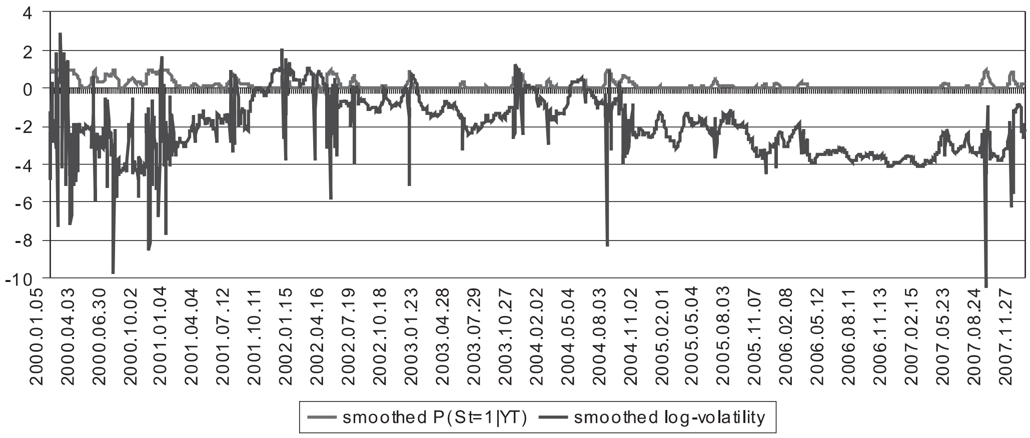

Figure 11. Smoothed probabilities $\Pr(S_t = 1|\Psi_T)$ and log-volatilities for MSSV(μ, φ, σ)

Table 5

Estimates of the regime-specific and unconditional characteristics of the log-volatility process

Model	Characteristics							
	p_0	p_1	$E(x_t\|S_t = 0)$	$E(x_t\|S_t = 1)$	$E(x_t)$	$Var(x_t\|S_t = 0)$	$Var(x_t\|S_t = 1)$	$Var(x_t)$
MSSV(μ)	0.665	0.335	-2.935	-0.788	-2.217	1.990	1.995	3.018
MSSV(φ)	0.639	0.361	-2.596	-1.009	-2.023	2.956	1.310	2.943
MSSV(σ)	0.942	0.058	-1.957	-1.957	-1.957	3.752	65.232	7.334
MSSV(μ, φ, σ)	0.830	0.170	-2.023	-2.082	-2.033	3.580	8.015	4.334

In the case of the model in which either the intercept or the autoregressive parameter is regime-dependent the results seem to display clear evidence of there occurring structural shifts in the modelled series. In the first model (i.e. MSSV(μ)) the two regimes are distinguished solely in terms of the state-specific volatility means (as the point estimates of state-conditional variances of the log-volatility process are almost equal), whereas in the MSSV(φ) model – also in terms of the state-specific volatility variances. When only the volatility intercept is made regime-dependent, then the first regime (denoted as '0') is characterized by a lower volatility mean. If only the autoregression parameter is allowed to switch between the states, the first regime is not only the one of a lower volatility mean but also of a greater volatility variance. Nevertheless, both models are consistent as regards the unconditional log-volatility mean and variance as well as the ergodic probabilities of each of the two regimes. The point estimates of the latter indicate that for approximately two thirds of the analyzed sample the underlying Markov chain stays in the first regime and for the remaining part of the sample – in the second one. The results obtained with these two models are intuitively appealing as the regime shifts seem to occur only three times throughout the analyzed period (the first one at the very beginning of the sample, the second one in the middle of 2001, and the last one about September 2004). The smoothed probabilities $\Pr(S_t = 1|\Psi_T)$ displaying no 'peaks' clearly imply that the distinguished sub-periods hold without any abrupt and short switches between the states of the system.

It is worth noticing that both constructions yield very close results as regards separation between the high- and low-volatility periods, indicating a somewhat 'smooth' transition (over the first half of 2001) from the less to more volatile period (see Figure 8-9), and then a rather 'steep' switch back to the more 'sedate' regime (around September 2004). These results coincide with a general profile of the economy of the time, suffering from both economic and financial downturn. Deterioration of the Polish banking sector over the period from 2001 to 2003, incidental to a general decline in the economy's growth rate, was accompanied by the WIBOR1M interest rates following a regular downward trend (from about 20% to 5%). However, once the economy rebounded in 2004, the interest rates have stabilized (in the second half of 2004) and featured markedly lower variability, henceforth.

The remaining two models, i.e. MSSV(σ) and MSSV(μ, φ, σ), provide conclusions that no longer remain in accordance with the former ones. The states are distinguished only in terms of the regime-specific log-volatility variances. In the case of the MSSV(σ) model this results from the model's construction itself, as allowing only the volatility of volatility parameter to switch between the regimes does not allow differences in the state-conditional log-volatility means. However, even if all of the volatility parameters are made regime-dependent and so the above restriction no longer holds, the volatility mean level is almost equal over the regimes. On the other hand, both of the models imply that the second regime is markedly more volatile than the first one. Yet, it is the state in which the unobserved Markov process stays far less frequently than in the second state (about 17% of the sample, according to the MSSV(μ, φ, σ) model, and only about 6% – according to the MSSV(σ) model). What is more, the series of the smoothed probabilities, $\Pr(S_t = 1|\Psi_T)$, feature evident irregularities (see Figure 10-11), so that, contrary to the previous switching specifications, no economic reasoning underlying that case could be found by the authors.

Finally, we compare the performance of each of the estimated models with respect to the smoothed log-volatilities. Specifically, we are interested in the differences between the estimates of the $\ln h_t$'s ($t = 1, 2, \ldots, T$) obtained with a non-switching specification (the BSV model) and those from each of the models allowing for discrete changes in the parameters' values. In each of Figures 12-15 we also plot the modelled series of AR-residulas, z_t, for WIBOR1M (for the sake of the pictures, z_t's are shifted by five) as to visually assess the relevance of the $\ln h_t$'s smoothed estimates with the volatility 'observed' in the data itself.

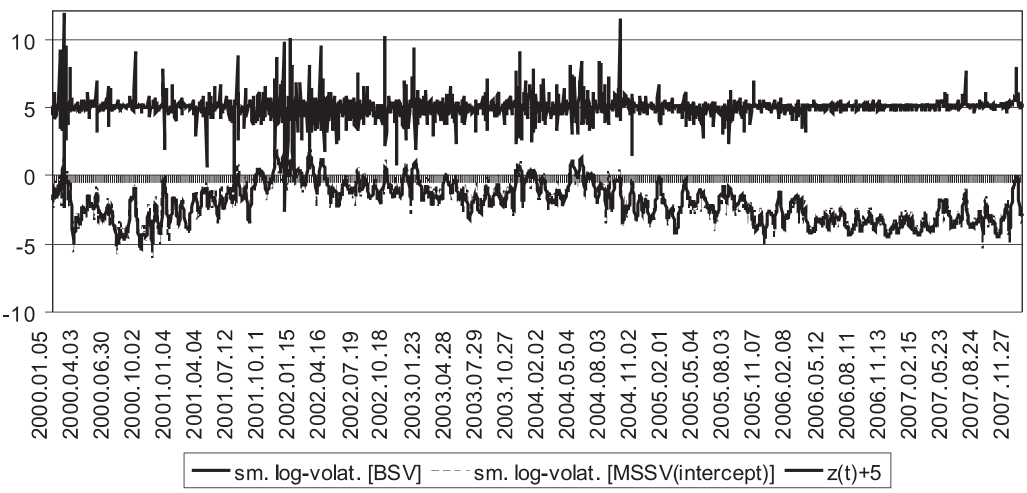

Figure 12. Smoothed log-volatilities for BSV and MSSV(μ), and AR(1)-residuals (shifted by 5) for WIBOR1M

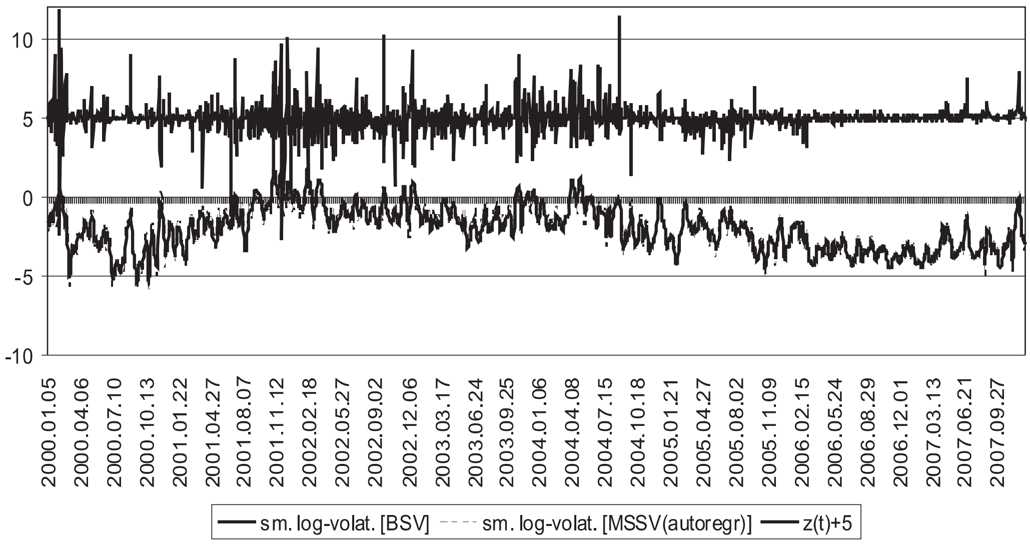

Figure 13. Smoothed log-volatilities for BSV and MSSV(φ), and AR(1)-residuals (shifted by 5) for WIBOR1M

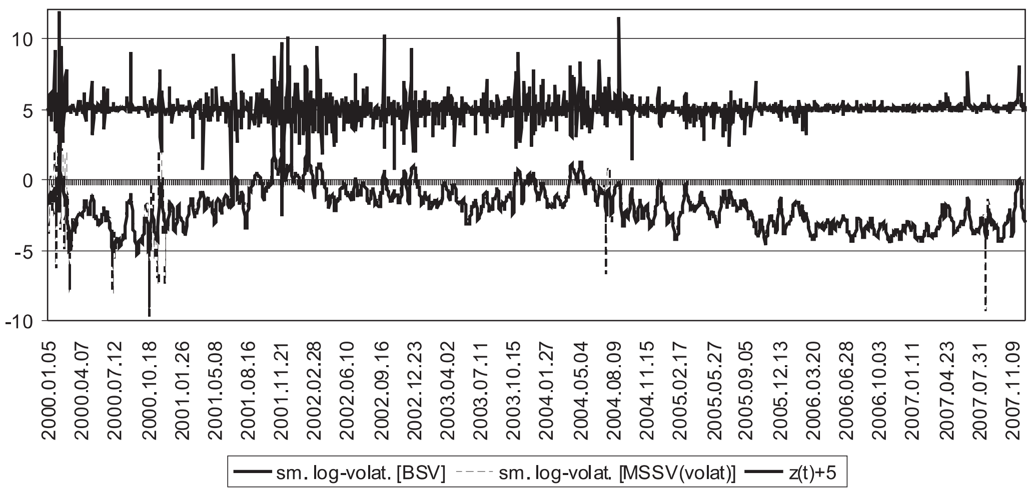

Figure 14. Smoothed log-volatilities for BSV and MSSV(σ), and AR(1)-residuals (shifted by 5) for WIBOR1M

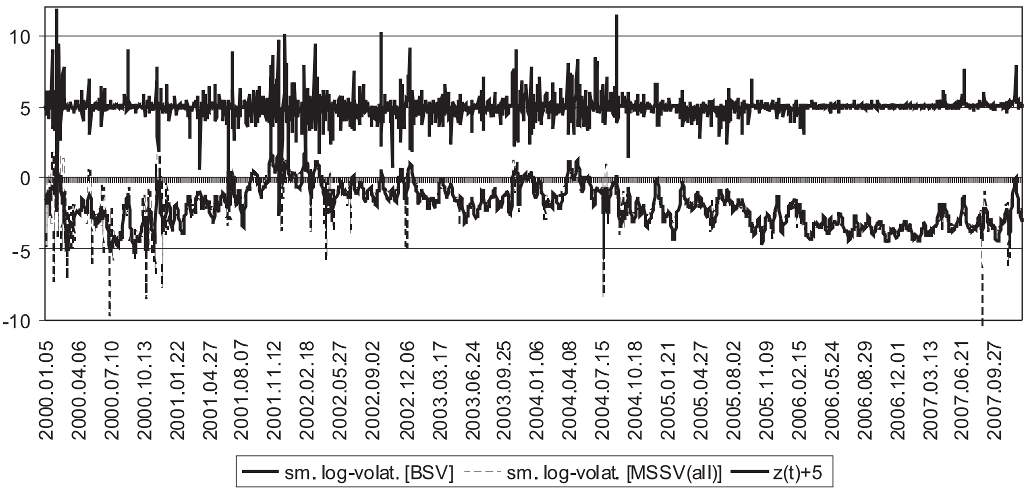

Figure 15. Smoothed log-volatilities for BSV and MSSV(μ, φ, σ), and AR(1)-residuals (shifted by 5) for WIBOR1M

If either the volatility intercept or the autoregressive parameter is allowed to switch over the regimes, the smoothed log-volatilities from the Basic SV model and the switching specification are roughly the same. However, this is no longer the case if we consider the MSSV(σ) and MSSV(μ, φ, σ) models, in which for the most part of the sample the estimates of $\ln h_t$'s obtained with the regime-switching model are smoother than the ones from the BSV construction and for the remaining part – far more volatile. On should note that these short intervals of extreme volatility do not always 'correspond well with' the modelled series and so the two models become somewhat less comprehensible (in terms of formulating economic reasons underlying the obtained results).

5. CONCLUSIONS

In the paper we present a particular generalization of a well-known stochastic volatility model. We allow for discrete changes in the volatility parameters' values in order to account for potential structural breaks in the modelled time series and hence different states of the economy. The regime changes are driven by a two-state homogenous Markov chain, as suggested by Hamilton in [11]. The resultant Markov-switching SV model is fitted via the QML procedure proposed by Smith in [18] to the AR(1)-residuals for the 1-month Warsaw Interbank Offered Rate interest rates over the period from January 3, 2000 to December 18, 2007. Four switching specifications are considered – three of them with a single volatility parameter being regime-dependent and the one in which all of the volatility parameters are allowed discrete shifts.

It is found that all of the models incorporating switching mechanism are preferred to a simple SV specification in the light of the information criteria as well as descriptive statistics of the standardized residuals. Since the estimates of the switching parameters are clearly different over the regimes we conclude that structural changes do occur within the modelled series and, based on the information criteria, these should be accounted for. However, different specifications of the MSSV model may lead to different conclusions as regards the manner in which the regimes switch and the way the volatility process evolves.

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PROCESY MARKOV SWITCHING SV W MODELOWANIU ZMIENNOŚCI FINANSOWYCH SZEREGÓW CZASOWYCH

Streszczenie

W artykule prezentujemy przełącznikowe modele stochastycznej zmienności (ang. *Markov Switching Stochastic Volatility*, MSSV) w kontekście ich wykorzystania na gruncie ekonometrii finansowej. Modele te stanowią pewne uogólnienie powszechnie znanych w literaturze modeli stochastycznej zmienności (SV), w której wartości parametrów równania specyfikującego warunkową wariancję procesu generującego obserwacje mogą ulegać skokowym zmianom w czasie. Służy to konstrukcji modelu pozwalającego uwzględniać potencjalne zmiany strukturalne, mające miejsce w okresie, z którego pochodzą analizowane dane finansowe. Przyjmuje się, iż przełączanie między różnymi stanami (reżimami) następuje zgodnie z jednorodnym łańcuchem Markowa. W pracy ograniczono się do przypadku, gdzie liczba reżimów wynosi dwa. Do oszacowania przedmiotowych modeli wykorzystujemy podejście oparte na metodzie *quasi*-największej wiarygodności (ang. *Quasi-Maximum Likelihood method*, QML), zaprezentowanej w pracy [18]. W części empirycznej dokonujemy analizy szeregu czasowego wysokości oprocentowania 1-miesięcznych lokat międzybankowych, WIBOR1M. Estymacji poddajemy cztery modele z rodziny MSSV oraz podstawowy model stochastycznej zmienności (ang. *Basic Stochastic Volatility*, BSV). Porównania w zakresie dobroci dopasowania wyżej wymienionych konstruktów do danych empirycznych dokonujemy poprzez kryteria informacyjne oraz charakterystyki opisowe standaryzowanych reszt.

Słowa kluczowe: modele stochastycznej zmienności, modele przełącznikowe Markowa, metoda quasi-największej wiarygodności

MARKOV SWITCHING SV PROCESSES IN MODELLING VOLATILITY OF FINANCIAL TIME SERIES

Summary

This paper presents a Markov Switching Stochastic Volatility model (MSSV) as a specification of potential use in financial econometrics. The model may be viewed as a specific generalization of a well-known SV construction, that allows the parameters of the conditional volatility equation to switch between a predetermined number of states (regimes). The switching mechanism is driven by a homogenous discrete Markov chain. Without significant loss of generality we restrict our analysis to two regimes only. Then we concentrate on the estimation procedure of a MSSV model, based on the Quasi-Maximum Likelihood approach presented by Smith in [18]. In order to calculate the quasi-log-likelihood function we consider a linear state-space representation of the MSSV model and employ a combination of the Kalman filter and Hamilton's filter procedures. Finally, four MSSV models and a standard SV model are estimated and compared in terms of goodness of fit to the 1-month WIBOR interest rates.

Key words: Markov switching, stochastic volatility, quasi-maximum likelihood estimation